EFR summary

Mathematics and Game Theory

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Lectures 1 to 5

Weeks 1 to 5

Details

Subject: Mathematics and Game Theory

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Mathematics & Game Theory – Lecture 1, week 1

Game Theory

Definition

Game - a situation where several players take actions that affect the other. A decision made by one player can have an impact on the payoffs of the other players.

Criteria to describe a game mathematically:

- A set of players (e.g., two players in a game of Rock, Paper, Scissors).
- A complete description of the available actions for each player.
- Information structure—what players know at the time of making a decision (e.g., whether they observe each other's moves).
- Outcome function—a rule determining how players' actions lead to different outcomes.
- Players' preferences over outcomes—expressing their ranking or desirability of different possible results (e.g., both players prefer to win).

Basic assumptions:
The game is common knowledge:

- each player knows the game being played,
- each player knows that every other player knows the game being played,
- each player knows that every other player knows that each player knows the game being played, etc.

Rationality:

- Players are assumed to act rationally, meaning they always select the action that maximizes their payoff (utility).

Forms of Games

Mathematical models represent games in two primary forms: **Extensive Form** and **Normal Form**.

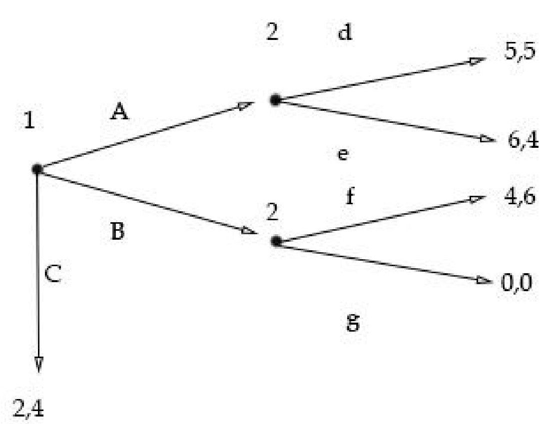
Extensive Form

Definition

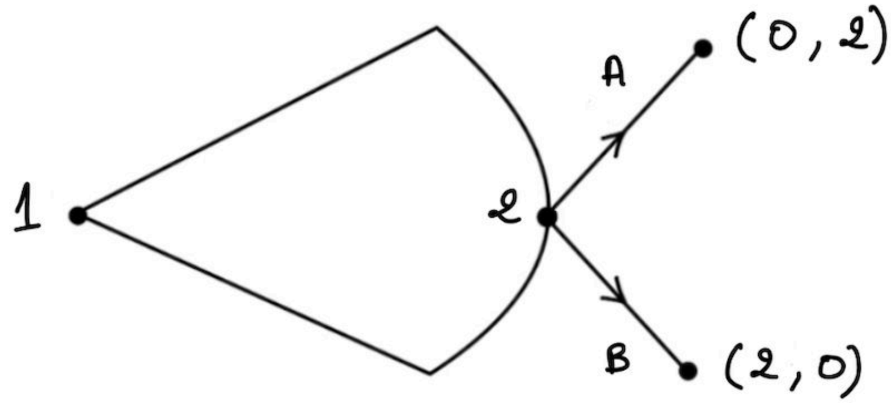
The **extensive form** represents a game as a **tree structure**, detailing the sequence of decisions made by players and the possible payoffs. This format allows for representation of **imperfect information** (where players may not have complete knowledge of past actions).

Information Set:

- Nodes and branches represent decision points and possible actions, respectively.
- Information sets group decision nodes that are indistinguishable to a player (depicted by dashed lines).
- Perfect information games (e.g., Chess) have no dashed lines since all actions are observable.
- Imperfect information games (e.g., Poker) include dashed lines, indicating hidden actions.

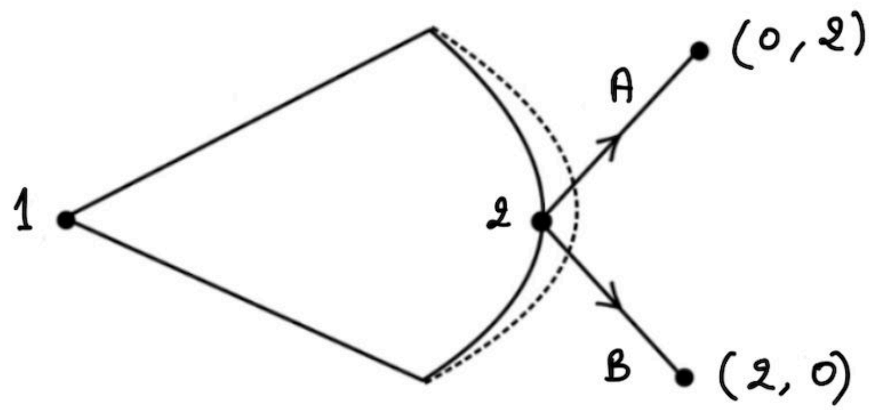


Games featuring an infinite set of actions can be visually represented using an arc connected by two branches:



In the accompanying figure, Player 1 has an unlimited number of possible actions. Consequently, Player 2 faces an infinite number of decision points and must choose between strategy "A" or strategy "B" in response.

In cases of imperfect information—where Player 2 cannot observe Player 1's action—all of Player 2's decision points merge into a single information set. This scenario is depicted using a dashed line, illustrating the uncertainty in Player 2's decision-making process.



Key points:

- There is always one initial node
- Every branch has a label
- Every terminal node has a payoff rector
- At every non-terminal node, indicate the player

Tree rules:

- Tree rule 1: Every node is a successor of the initial node, and the initial node is the only one with this property.

- Tree rule 2: Each node except the initial node has exactly one immediate predecessor. The initial node has no predecessors.
- Tree rule 3: Multiple branches extending from the same node have different action labels.
- Tree rule 4: Each information set contains decision nodes for only one of the players.
- Tree rule 5: All nodes in a given information set must have the same number of immediate successors and they must have the same set of action labels on the branches leading to these successors.
- Tree rule 6 (perfect recall): players remember their own past actions as well as any other events that they have observed.

Normal form

Definition

The **normal form** (also called strategic form) represents a game using a payoff matrix, making it useful for identifying dominant strategies and Nash Equilibria.. The normal-form representation of a game includes all perceptible and conceivable strategies, and their corresponding payoffs, for each player.

Components of Normal Form

In a normal form, the set of players can be depicted inside brackets, with each player separated by a comma. , e.g., {A, B, C} or {1, 2, 3}.
Strategy spaces: Each player has a set of possible strategies, denoted as : $S_1, S_2, S_3, \ldots, S_n$ for each player in the game.
Payoff functions: The utility function u_i(s) defines player i's payoff for each strategy profile, can be depicted as: $U_1, U_2, U_3, \ldots, U_n$.

Infinite strategy spaces

In normal-form games with infinite strategy spaces, players are not confined to a limited set of predefined actions or choices. Instead, they can select from an infinite number of possible strategies, typically represented by continuous variables.

Example:

In the mid-1800s, an economist developed a model to describe how two firms compete in a market by deciding how much to produce. Suppose that firms A and B manufacture an identical product, meaning consumers are indifferent to purchasing from either firm.
Each firm selects its production level simultaneously and independently. Let $q_A \geq 0$ represent the quantity produced by Firm A (in thousands of units) and $q_B \geq 0$ represent Firm B's production. The total market supply is then given by:

$$q_A + q_B$$

The price of the product is determined by a simple demand function:
$p = 1000 - (q_A + q_B)$ Additionally, each firm incurs a production cost of €150 per thousand units produced.

$N = \{A, B\}$

$S_A = S_B = [0, \infty)$

$U_A(q_A, q_B) = 1000 - (q_A + q_B).q_A - 100q_A$

$U_B(q_A, q_B) = 1000 - (q_A + q_B).q_B - 100q_B$

Normal Form Games: Matrix Games

For **two-player** and **three-player** normal-form games with finite strategy spaces, the game can be effectively described using a **payoff matrix**. Each row represents a possible strategy for one player, while each column (or additional dimension for three-player games) represents the strategies of the other player(s). The matrix entries contain the corresponding payoffs for each player based on their chosen strategies.

This matrix representation provides a clear and structured way to analyze strategic interactions, determine best responses, and identify equilibrium solutions such as Nash equilibria.

Example:
Three colleagues are deciding whether to go to a café. Each of them prefers to go only if exactly one other person also decides to go. If only one person goes alone, they will feel awkward, and if all three go, the café will be too crowded and uncomfortable.

The payoffs for each individual are as follows:

- If you stay home, your payoff is 3.

- If you go to the café alone, your payoff is 1.
- If you go to the café with one other colleague, your payoff is 4.
- If all three go together, your payoff is 0.

Draw the normal-form matrix.
Player 3:C

1\2	B	H
B	0,0,0	4,3,4
H	3,4,4	3,3,1

Player 3:H

1\2	B	H
B	4,4,3	1,3,3
H	3,1,3	3,3,3

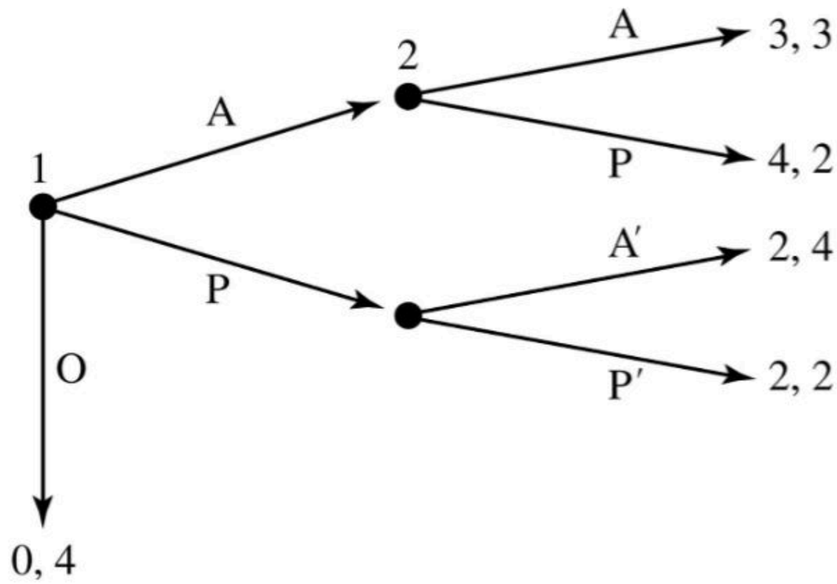
Extensive form vs Normal form

Feature	Extensive Form	Normal Form
Representation	Game tree (nodes & branches)	Payoff matrix
Best suited for	Sequential games, imperfect information	Simultaneous games, clear payoffs
Conversion	Can be converted to normal form in one unique way	Can be represented in multiple extensive forms

- Converting an extensive-form game to normal form results in a single unique representation.
- Converting a normal-form game to extensive form can be done in multiple ways.
- Extensive form is preferred for games involving a strict sequence of actions or imperfect information.

Example:

Draw the normal-form matrix of this game.



1\2	AA’	AP’	PA’	PP’
A	3,3	3,3	4,2	4,2
P	2,4	2,2	2,4	2,2
O	0,4	0,4	0,4	0,4

Mathematics & Game Theory – Lecture 2, week 2

Static Games

Static games refer to strategic interactions where all players make their decisions simultaneously, without knowing what choices the other players have made. Since decisions are made at the same time and independently, players must anticipate their opponents' choices based on available information. These types of games are best represented using normal form models, which display the players, their possible strategies, and the corresponding payoffs in a structured format.

Belief: In a static game, each player forms an expectation about what strategies the other players might choose. This expectation is known as a belief, which is represented mathematically as a probability distribution over the possible strategies of opponents. Essentially, a belief quantifies how likely a player thinks each possible move of their opponents is.

Mixed strategy: occurs when a player does not commit to a single specific move but instead chooses among available strategies according to a probability distribution. This means that rather than always selecting the same action, the player randomizes their choice based on assigned probabilities.

Pure strategy: a mixed strategy that assigns all probability to one strategy. Note: a pure strategy is also a mixed strategy, but not necessarily the other way around.

Expected payoff: the "weighted average" payoff a player gets for a given strategy profile.

Strict Dominance

In game theory, a **strictly dominated strategy** is a strategy that is always worse than another available strategy, regardless of how the opponents play. This means that there exists another strategy—either pure or mixed—that provides a higher payoff in every possible scenario. Because strictly dominated strategies are always suboptimal, rational players will never choose them.

Method for checking whether a strategy is dominated:

- First, decide whether it is dominated by another pure strategy:

see if there is another **pure strategy** that always yields a higher payoff than the one in question, regardless of what the other players do.
If such a better pure strategy exists, then the given strategy is strictly dominated and should never be played.

- Otherwise, check whether it is dominated by a mixed strategy:

1. Look for <u>alternating patterns</u> of large and small numbers in the payoff matrix, which may indicate that a mixed strategy can provide a consistently better outcome.

2. Write down the corresponding equations.
3. Solve these equations to determine if a mixed strategy dominates the given strategy.

1\2	A	B
M	3,4	7,0
S	2,0	5,4

Determine which strategies are dominated:
Player 1: S is dominated by M
Player 2: No dominated strategy

Iterated Dominance

- In strategic decision-making, a **dominated strategy** is one that always results in a worse outcome than another available strategy, regardless of what the opponents choose. Since rational players aim to maximize their payoffs, they will never play a dominated strategy.
- **Iterated dominance**: repeatedly eliminating dominated strategies from the game until only the most rational choices remain.
    - Delete all of the dominated strategies for each player.
    - R1 are the strategy profiles that remain.
    - Delete from R1 any strategies that are dominated in this reduced game.
    - R2 are the strategy profiles that remain.

Note: The order in which strategies are eliminated **does not matter** as the final result will always be the same.

Rationalizable strategies: the set of strategies that survive iterated dominance, By applying iterated dominance, players can simplify strategic decision-making and focus only on rational choices that maximize their expected payoffs.

Example:

a. Determine the set of rationalizable strategies in the following game.

1\2	A	B	C
D	3,7	0,5	5,5
E	4,4	0,0	2,6
F	2,2	4,6	3,4

Solution:

the rationalizable strategies in the game :
Player 1: {D,E,F}
Player 2: {A,B,C}

Best Response

A **best response** is the strategy that provides the highest possible payoff for a player, given the strategies chosen by the other players. In other words, it is the optimal decision a player can make when assuming that their opponents' strategies are fixed.

Example:

a. *Determine* $BR_{1}(\frac{1}{3}, \frac{1}{2}, \frac{1}{4})$

1\2	E	F	G
A	3,7	0,5	5,5
B	4,4	0,0	2,6
C	2,2	4,6	3,4

Solution:

- If player 1 selects A, his payoff would be: $\frac{1}{3} * 3 + \frac{1}{2} * 0 + \frac{1}{4} * 5 = 2.25$
- If player 1 Selects B, his payoff would be: $\frac{1}{3} * 4 + \frac{1}{2} * 0 + \frac{1}{4} * 2 = 1.83$
- If player 1 Selects C, his payoff would be: $\frac{1}{3} * 2 + \frac{1}{2} * 4 + \frac{1}{4} * 3 = 3.42$

Hence, the best response for player 2 would be to select strategy A.

Nash Equilibria

Nash equilibrium: strategic outcome in which no player can improve their payoff by unilaterally changing their strategy, assuming that the other players keep their strategies unchanged. In other words, at a Nash equilibrium, every player is playing their **best response** to the strategies chosen by the others.

Nash equilibria can be:

- Pure-strategy (by removing rows and/or columns that are dominated)
- Mixed-strategy:
- involves players randomizing their choices based on a probability distribution.
- Set up equations that represent the expected payoffs for different strategies..
- Solve these equations to determine the probabilities with which players should mix their strategies.
  - Every game with a finite number of players and a finite strategy space has at least one Nash equilibrium.

Compute the pure-strategy Nash equilibria:

1\2	M	N	P
K	6,7	4,8	0,5
L	9,4	4,2	6,3
J	8,6	5,5	6,7
S	4,5	8,6	4,4

3 Pure Nash Equilibria: (L,M), (J,P), (S,N)

Dynamic Games

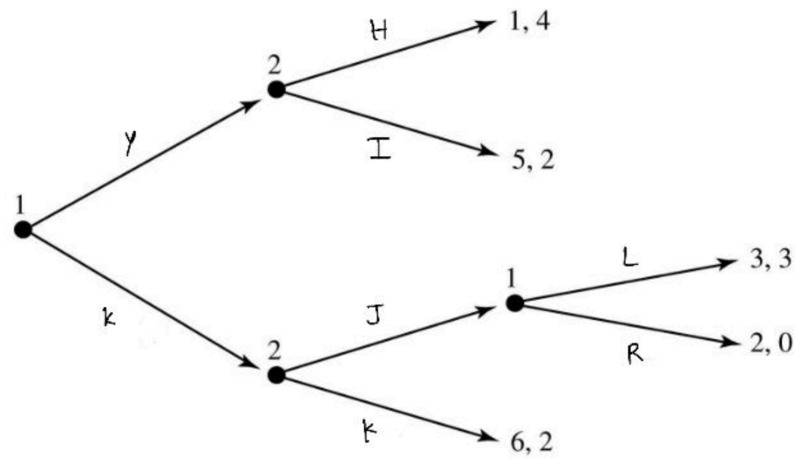
Backward induction

In **dynamic games**, players make decisions sequentially according to a predetermined timeline. These games are best represented using the extensive form, which captures the order of moves and possible choices at each stage.

Backward induction: a method used to solve such games by working in reverse, starting from the final decision and moving backward to the beginning. At each step, suboptimal choices are eliminated, ensuring that only rational decisions remain.

This approach is only applicable to games with **perfect information**, where each player is fully aware of all previous moves when making a decision.

Solve the game by using backward induction:



(KL, HJ)

Subgames

A **subgame** is a distinct part of an extensive-form game that satisfies the following conditions:

- It begins at a single **initial node**, which must be the only node in its information set.
- If a node is included in the subgame, all of its **successor nodes** must also be included.
- If any node from a particular **information set** is in the subgame, then all nodes in that information set must be part of the subgame as well.

Types of Subgames

Proper subgame: a subgame that starts from nodes other than the initial node.
Minimal proper subgame: a proper subgame that does not contain another proper subgame.

Subgame perfect Nash equilibria

Subgame perfect Nash equilibrium (SPNE): a strategy profile in which players play a **Nash equilibrium** in every subgame of the original game. This ensures that players make optimal choices at every stage, not just in the overall game but also within any possible subgame.

- SPNE can be both pure-strategy and mixed-strategy.
- Finding SPNE:
    1. Identify the minimal proper subgame and determine a Nash equilibrium within it.
    2. Replace this subgame with the payoff vector corresponding to the selected equilibrium.
    3. Repeat the process for the reduced game until reaching the initial decision node.
    - SPNE is most effectively applied in perfect information games using backward induction, ensuring rational decision-making from the last move to the first.

Mathematics & Game Theory - IBEB - Lecture 3, week 3

Integration

Definition

Integration is the reverse process of differentiation, often called taking the antiderivative. It involves determining the original function given its derivative, essentially reconstructing the function from its rate of change.

Types of Integrals

There are two types of integrals, namely definite and indefinite integrals.

Indefinite Integrals

Definition

The function F(x) is called an indefinite integral of f(x) if f(x)=F'(x). In other words, integration is the reverse process of differentiation.

- Indefinite integrals do not have specified limits and are expressed as:

$$\int f(x)dx = F(x) + C$$

- where C is an arbitrary constant. This constant arises because differentiation of $F(x) + C$ still results in f(x), since the derivative of a constant is zero.
- Indefinite integrals are also known as antiderivatives.

Applications for integrals

1. Computing areas in graphs
    a. Integration is used to calculate the area under a curve, which is particularly useful in geometry, physics, and economics.
2. Computing probabilities
    a. In probability theory, integration is used to find the probability of an event occurring within a specific range on a probability density function (PDF) curve.
3. Consumer and Producer Surplus
    a. When demand and supply functions are not linear, integration helps determine consumer and producer surplus by computing the exact area between the curves.

Important Integrals

Here is a list of important integrals to be used in this course:

- $\int x^b dx = \frac{1}{b+1}x^{b+1} + C$, where b≠-1
- $\int e^{bx} dx = \frac{e^{bx}}{b} + C$ provided that b≠0

- $\int \frac{1}{x} dx = ln|x| + C$

- $\int a^x dx = \frac{a^x}{ln\, a} + C$, where a>0 and a≠1.

General rules for integrals

Let b≠0 be a constant.

- $\int bf(x)dx = b\int f(x)dx$

- $\int (f(x) \pm g(x))dx = \int f(x)dx \pm b\int g(x)dx$

Example: $\int \frac{1}{2}x + 5e^{x} dx = \frac{1}{2}\int x\ dx + 5\int e^{x} dx$

Definite Integrals

Definition

A **definite integral** is an integral with specified lower and upper limits, which define the interval over which the function is integrated. Given a function f(x) over the interval [a,b], the definite integral is written as: $\int_a^b f(x)dx$.

Unlike indefinite integrals, definite integrals evaluate to a numerical value rather than a function. If F(x) is an antiderivative of f(x), meaning F'(x)=f(x), then the definite integral is calculated using the **Fundamental Theorem of Calculus**:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where F(x) is any indefinite integral of f(x).

How to Evaluate a Definite Integral:

1) Find the indefinite integral F(x) of f(x).
2) Evaluate F(x) at the upper limit b.
3) Evaluate F(x) at the lower limit a.
4) Subtract: F(b)-F(a).

Properties

Let $f(x)$ be a continuous function on an interval containing c,d,e,m and n, and let βbe a constant. The following properties hold for definite integrals:

1. $\int_m^n f(x)dx = -\int_n^m f(x)dx$
2. $\int_c^c f(x)dx = 0$
3. $\int_m^n \beta f(x)dx = \beta\int_m^n f(x)dx$
4. $\int_m^n [f(x) + q(x)]dx = \int_m^n f(x)dx + \int_m^n q(x)dx$
5. $\int_c^d f(x)dx = \int_c^e f(x)dx + \int_e^d f(x)dx$

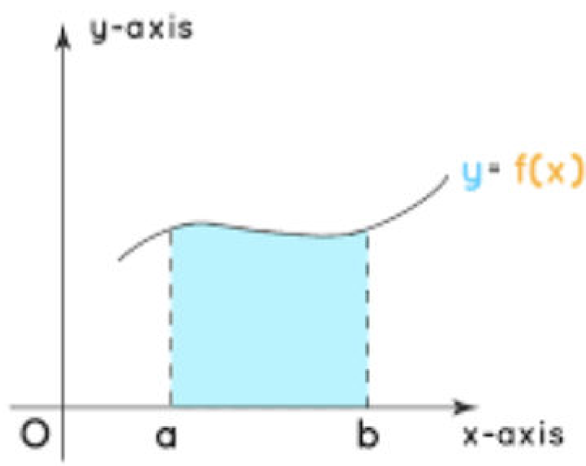
Example: Compute the integral $\int_0^2 x^2 + 5\,dx$

$$= \left[\frac{x^3}{3} + 5x\right]_0^2$$

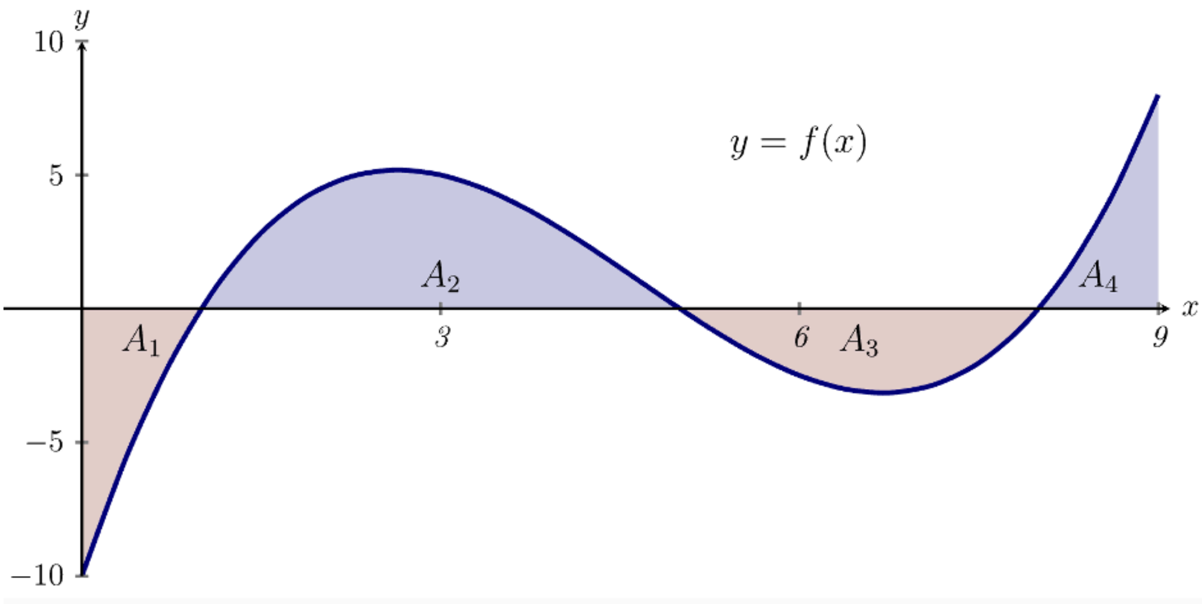
$$= \left(\frac{2^3}{3} + 5 * 2\right) - \frac{0^3}{3} + 5 * 0 = \frac{8}{3} + 10$$

Areas and Definite Integrals

Definite integrals can be used to compute the area between functions or a function and a line. As in this case:



The area represented by blue can be computed by using the formula $\int_a^b f(x)dx$, if f(x)≥0 for every x and $-\int_a^b f(x)dx$ if f(x)≤0 for every x.



Derivative of an Integral

Let us denote $A(x) = \int_{f(x)}^{q(x)} h(t)dt$. After solving the definite integral, we will eventually reach $A(x) = B(t)_{f(x)}^{q(x)} = B(q(x)) - B(f(x))$, where B'(t)=b(t). The integral of this function will be written in terms of x, therefore we can also find its derivative. Thus, A'(x) is given by the following formula:

$$\frac{d}{dx}A(x) = \frac{d}{dx}\int_{f(x)}^{q(x)} b(t)dt = \frac{d}{dx}(B(q(x)) - B(f(x))) = b(q(x))g'(x) - b(f(x))f'(x)$$

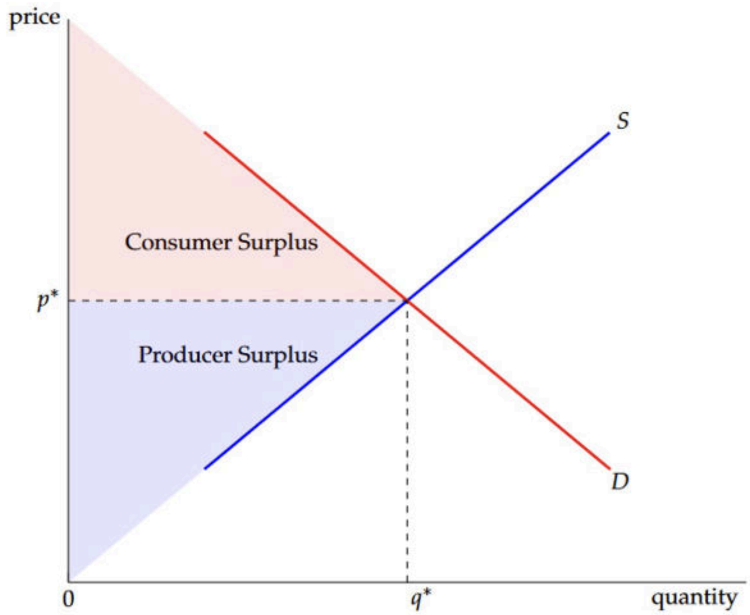
In short, $A'(x) = h(g(x))g'(x) - h(f(x))f'(x)$.

Consumer and Producer Surplus

The consumer and producer surplus can be determined by identifying the intersection of the demand curve and the supply function along with the Oy-axis.

To calculate these areas using definite integrals, we first need to find the equilibrium point (P*,Q*), where the demand and supply functions intersect. Once this point is identified, we can apply the formula for finding the area between two curves.

Let D(Q) represent the demand function and S(Q) represent the supply function. The consumer surplus is the area between the demand curve and the equilibrium price, while the producer surplus is the area between the equilibrium price and the supply curve. These values can be determined using integration.



Consumer Surplus

We have the following formula for determining consumer surplus (CS):

$$CS=\int_{0}^{Q*}(D(Q) - P*)dQ$$

Producer Surplus

We have the following formula for determining producer surplus (PS):

$$PS=\int_{0}^{Q*}(P*- S(Q))dQ$$

The Substitution Rule

Indefinite Integrals

The substitution rule is a powerful technique in integral calculus that simplifies the process of evaluating indefinite integrals. It states that if we define u=g(x), then the derivative g’(x)dx can be expressed as du. This substitution transforms the given integral into a more manageable form while preserving the original integral’s solution. Mathematically, the substitution rule is expressed as:

$$\int f(g(x))g^{'}(x)dx = \int f(u)du = F(u)$$

where F’(u)=$f(u)$

Once the indefinite integral F(u) is determined, we substitute u back with g(x), yielding the final result:

$$\int f(g(x))g^{'}(x)dx = F(g(x))$$

Definite Integrals

The substitution rule for definite integrals works similarly to that for indefinite integrals, with the key difference that the limits of integration must also be adjusted based on the substitution.

$$\int_{m}^{n} f(g(x))g^{'}(x)dx = \int_{g(m)}^{g(n)} f(u)du$$

Key Steps for Applying the Substitution Rule in Definite Integrals

1. Choose a substitution: Let u=g(x), then compute du=g’(x)dx.
2. Adjust the limits of integration: Transform the original limits x=m into u-values by substituting them into u=g(x).
3. Rewrite the integral in terms of u: Replace all occurrences of x with u, including the differential dx.
4. Evaluate the new integral: Compute the definite integral in terms of u and evaluate it using the transformed limits.

Mathematics & Game Theory - IBEB - Lecture 4, week 4

Indefinite Integrals

Infinite Integrals

Definition

An **improper integral** is a type of definite integral where at least one of the limits of integration is either **infinite** (-∞ or +∞) or the function being integrated is **undefined** at one or more points within the limits. This means that the usual methods for evaluating definite integrals do not directly apply, and special techniques must be used.

Forms of an Improper Integral

An improper integral has one of the following forms:

1. $\int_{m}^{\infty} f(x)dx$
2. $\int_{-\infty}^{n} f(x)dx$

Method

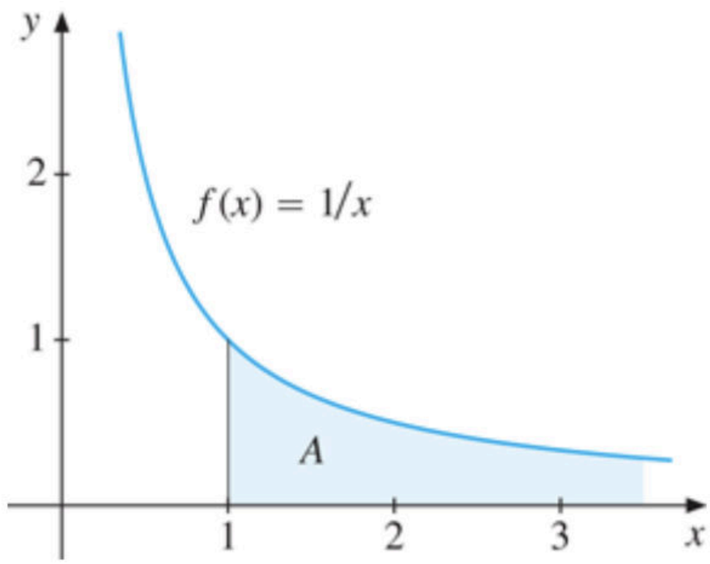
To solve an improper integral, the use of limits is required. Thus, by transforming -∞ and +∞ into limits, we reach the following forms of the improper integrals mentioned before. To solve an improper integral, make sure to substitute -∞ or +∞ with a constant and continue solving:

1. $\int_{m}^{\infty} f(x)dx = \int_{m}^{n} f(x)dx$
2. $\int_{-\infty}^{n} f(x)dx = \int_{m}^{n} f(x)dx$

Convergence and Divergence

- If the limit of the improper integral exists and results in a finite number, the integral is said to converge.
- If the limit does not exist or tends to infinity, the integral is divergent and does not have a finite value.

Example:
Let A represent the area beneath the curve of the function f(x)=$\frac{1}{x}$ over the interval [B,∞], where B is a positive constant. Find the value of A, if it exists.



$$\int_{B}^{b} \frac{1}{X} dx = [ln|x|]_{B}^{b} = ln(b) - ln(B)$$

$$\int_{B}^{\infty} \frac{1}{X} dx = \lim_{b \to \infty} \int_{B}^{b} \frac{1}{x} dx = \lim_{b \to \infty} (ln(b) - ln(B)0 = \infty$$

Since the limit evaluates to infinity, we conclude that the integral diverges, meaning A does not have a finite value. This implies that the area under the curve is infinite and cannot be determined as a finite number. The reason for this divergence is that the function f(x)=$\frac{1}{x}$ does not decrease rapidly enough toward zero as x approaches infinity.

Improper Integrals With Both - ∞ and ∞

When evaluating an improper integral with limits extending from $-\infty$ to ∞, it is important not to replace both infinities with a single constant. Since negative and positive infinity represent fundamentally different concepts, they cannot be treated as equivalent or directly compared. Instead, the integral should be approached by splitting it at a convenient point, such as zero or another finite value, and then evaluating each part separately.. Instead, refer to the steps below:

1. Split the integral on any real number (ex: 0)

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

2. Determine if the integral converges or diverges by checking if both integrals converge or diverge. If both integrals converge, then the integral converges. But if one or more diverges, the whole integral diverges.

Unbounded Functions

In some cases, it is possible to evaluate the definite integral of a function even if it is not defined at one of the integration limits. This can be demonstrated by examining the following two scenarios:

- If f(x) is not defined at its upper limit (x=c):

$$\int_{a}^{c} f(x)dx = \int_{a}^{i} f(x)dx$$

- If f(x) is not defined at its lower limit (x=d):

$$\int_{d}^{c} f(x)dx = \int_{i}^{c} f(x)dx$$

Improper Integrals (unbounded functions)

If the integral is continuous but not defined in both its upper and lower bounds, simply take a fixed constant within the interval and use it to split the function. For example, take a function f(x), which is continous on the interval (l,m). In this interval, x = l and x = m are not defined:

$$\int_{l}^{m} f(x)dx$$

To solve the function, take a fixed c from the interval (l, m) and define:

$$\int_{l}^{m} f(x)dx = \int_{l}^{m} f(x)dx + \int_{l}^{m} f(x)dx$$

Determine if the integral converges or diverges by checking if both integrals converge or diverge. If both integrals converge, then the integral converges. But if one or more diverges, the whole integral diverges.

Linear Programming

Definition

Linear programming is a mathematical technique used to determine the optimal outcome—such as maximizing profit or minimizing cost—within a system defined by linear equations and inequalities.

- The function being optimized is known as the objective function.
- The conditions that must be satisfied are called constraints.
- Together, the objective function and constraints form a linear program.

When solving a linear program, the goal is to find values of the variables that result in either the maximum or minimum possible value of the objective function while still satisfying all constraints.

Linear programming has numerous applications, particularly in economics, business, engineering, and logistics, where efficient resource allocation is essential.

Examples:

1. Determine the indefinite integral $\int \frac{2x ln(x^2+3)}{x^2+3} dx$

   Let $u = ln(x^2 + 3)$, then $du = \frac{2x}{x^2+3} dx$.

   So, we write the original integral as $\int u\, du = \frac{1}{2}u^2 + c$.

2. Compute $\int_{1}^{2} \frac{e^{\frac{x}{3}}}{x^2} dx$

   Let $u = \frac{x}{3}$. Then $du =- \frac{3}{x^2}dx$. So, $- \frac{1}{3}du = \frac{1}{x^2}dx$.

   Furthermore, if $x = 2$, then $u = \frac{3}{2}$, if $x = 1$, then $u = 3$.

   So, the integral is equivalent to:

$$\int_{3}^{\frac{3}{2}} - \frac{1}{3}e^{u}du = \left[- \frac{1}{3}e^{u}\right]_{3}^{\frac{3}{2}}$$

3. Give the optimal solution and the optimal value of the following linear programming problem:

Max $8x + 7y$

Subject to	$x\leq6$	(1)
	$x - y\geq - 2$	(2)
	$x + y\leq8$	(3)
	$x, y\geq0$	(4)

As shown in the above diagram, the shaded area is the feasible region set by the parameters (1 to 4). Set the objective $8x + 7y = c$, then $y =- \frac{8}{7}x + \frac{c}{7}$.

Maximising c is equivalent to maximising the intercept. So, it reaches the maximum at the intersection point of (1) and (3), i.e., point (6,2) with the optimal value $8 * 6 + 7 * 2 = 62$. When solving a linear programme like this, it may be helpful to first sketch the constraint boundaries in a graph and indicate the feasible region. Once this has been done, the objective can be sketched, and the optimum can be determined.

No Optimal Solutions

An optimal solution may not exist in the following cases:

1. Contradictory Constraints

   - If the given constraints contradict each other, they create an empty feasible region (i.e., no values satisfy all constraints simultaneously).

   - As a result, no solution exists for the linear program.

2. Unbounded Feasible Region

    - If the feasible region extends infinitely in the direction where the objective function is being optimized (maximized or minimized), then the solution is unbounded, meaning no finite optimal solution exists.

Infinitely Many Solutions

If the optimal solution lies along a line segment rather than at a single point, then there are infinitely many optimal solutions instead of just one.

This happens when multiple points along the segment provide the same optimal value for the objective function.

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Matrices

General form of a matrix

A matrix is a structured arrangement of numbers, used to represent mathematical data. Each individual number within the matrix is referred to as an element or entry. These elements are typically enclosed within square brackets [], though round brackets () may also be used in some cases. Matrices are usually represented using capital letters, such as A,B,C etc.

A matrix with m rows and n columns (m x n) has the following form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The small letters indicate elements of the matrix A. The indices (subscripts) indicate the location of each element. So, for example, a_{m1} indicates the element of A in the m-th row and the 1st column.

In a matrix, rows are arranged horizontally, while columns are arranged vertically. Therefore, a matrix with m rows and n columns has a dimension of m × n. When specifying the dimension of a matrix, always mention the number of rows first, followed by the number of columns.

An example for a matrix (2x3) would be:

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 0 & 0 \end{bmatrix}$$

Matrix addition & subtraction

Consider the following two matrices, both with m lines and n columns:

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 0 & 0 \end{bmatrix}$$

And

$$B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

In this case, by adding A and B we obtain the following result:

$$A + B = \begin{bmatrix} 7 & 5 & 2 \\ 5 & 2 & 0 \end{bmatrix}$$

In the same case, by subtracting A and B we obtain the following result:

$$A - B = \begin{bmatrix} 4 & -1 & 0 \\ 3 & -2 & 0 \end{bmatrix}$$

It is important to note that matrices must have the same dimensions in order to be added or subtracted. If two matrices do not have the same number of rows and columns, matrix addition or subtraction is not possible, and the result is undefined.

Scalar multiplication

By multiplying a scalar (say L) with a matrix, every element of the matrix is multiplied by that scalar. By considering the matrix A from above, this also works for negative numbers.

$$AL = \begin{bmatrix} 5L & 2L & 1L \\ 4L & 0L & 0L \end{bmatrix}$$

Matrix multiplication

Two matrices, A and B, can be multiplied together only if the number of columns in matrix A matches the number of rows in matrix B. This ensures that the elements align correctly for multiplication.

It is important to note that AB is usually different from BA.

The identity and zero matrices

In matrix algebra, two important special matrices are the Identity matrix and the Zero matrix. These matrices serve as the equivalents of 1 and 0 in standard arithmetic, respectively.

The Identity matrix is denoted with a capital "I" and is a square matrix. This means that it has as many rows as it has columns. There usually exists a subscript to indicate the dimensions (I_k means that the Identity matrix "I" has k rows and k columns). An identity matrix always has 1's in its diagonals and 0's everywhere else.

Example: $I_3 =$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

If a matrix is multiplied by an identity matrix, then we get the matrix itself again, regardless if it is multiplied from the left or from the right.

The Zero matrix is a matrix that only contains 0's. It can have a different number of rows and columns and may have subscripts that depict these rows and columns. Any matrix multiplied by the zero matrix garners the result of a zero matrix, though the dimension of the matrix may change.

Example: O_{2x3} =

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rules for matrix multiplication

The first multiplication rule is associativity: (AB)C = A(BC). This rule allows us to write powers with matrices (associativity).

The second and third rules are on distributivity:

A(B+C) = AB + AC (left distributivity)
(A+B)C = AC + BC (right distributivity)

Scalars can be multiplied anywhere.

$$(\alpha A)B = A(\alpha B) = AB\alpha$$

Some implications for real numbers do not have an equivalent for matrices:

- BC ≠ CB
- If BC = 0, this does not simply mean that B or C is equal to 0.
- If BC = BA, and A is not 0, do not imply that C = A

Matrix transposition

Definition

The transpose of an n × p matrix A, denoted by A^T or A', is a p × n matrix. In the transpose, the rows of the original matrix become the columns of the new matrix.
Example:

If B:

$$\begin{bmatrix} 1 & 6 \\ 6 & 0 \\ 2 & 4 \end{bmatrix}$$

Then B':

$$\begin{bmatrix} 1 & 6 & 2 \\ 6 & 0 & 4 \end{bmatrix}$$

Rules for transposition

Rules for transposition are as follows:

Rule #1: $\left(M^{'}\right)^{'} = M$

Rule #2: $(C + D)^{'} = C^{'} + D^{'}$

Rule #3: $(\sigma M)^{'} = \sigma M^{'}$

Rule #4: $(BC)^{'} = B'C'$

Symmetry

A matrix A is said to be symmetric if it is equal to its transpose. This means that flipping the matrix along its main diagonal (from the top-left to the bottom-right) does not alter its elements.

Systems of linear equations

Converting a system into matrices

Let us consider the following system of equations:

$$-2x + 5y = 7$$

$$8x + 2y = 10$$

The system can be rewritten as:

$$\left[\begin{array}{cc|c} -2 & 5 & 7 \\ 8 & 2 & 10 \end{array}\right]$$

"Linear in variables" means that each term in an equation involves only one variable raised to the first power. Therefore, systems of equations containing terms like x_1^2 *or* $x_1 . x_2$ (products of variables) are **not linear**. Only systems where each term is a simple variable (e.g., $x_1 . x_2$, etc.) without exponents or products of different variables can be expressed in matrix form.

Augmented matrix

Definition

An augmented matrix is formed by combining the coefficients of the variables from a system of equations along with the constant terms (often referred to as "free" terms). The matrix includes both the coefficients of each variable xi(where i=1,n) and the corresponding constants $b_1, b_2, ..., b_n$, representing the system in matrix form. This matrix is used to express the system of equations compactly for solving.

Elementary row operations

To solve systems of linear equations, an augmented matrix must be transformed into a simpler form. This transformation is achieved through elementary row operations, which include:

1. Swapping rows (interchanging two rows).
2. Multiplying a row by a nonzero scalar (scaling a row).
3. Adding a multiple of one row to another row (row replacement).

These operations help convert an augmented matrix into row echelon form, which follows these conditions:

- Any row consisting entirely of zeros appears at the bottom.
- Below each leading coefficient (pivot), all elements are zeros.
- The leading coefficient of a row is the first non zero value in that row.

By applying further row operations, the matrix can be transformed into reduced row echelon form, which satisfies the following additional conditions:

- The leading coefficient in each row is 1.

- Each leading coefficient is the only nonzero value in its column (all values above it are zero).

Gaussian elimination

Definition

Gaussian elimination is a technique for solving systems of linear equations by applying elementary row operations. This method systematically transforms a matrix into a staircase-like form (row echelon form), making it easier to find solutions.

Properties of a staircase

A matrix in staircase form follows these properties:

- The first nonzero entry in each row must be 1.
- All elements above and below this leading 1 must be 0.
- If two rows contain a leading 1, the row appearing first must have its leading 1 positioned in an earlier column.

Additionally, it is important to note that the solution obtained through Gaussian elimination is not always unique.

Solving systems – no solution & infinitely many solutions

A system of equations can have no solution or infinitely many solutions, depending on its structure after row echelon form is applied.

- A system is called inconsistent (no solution) if, after row operations, a row in the coefficient matrix becomes entirely zero, while the corresponding value in the constant column is nonzero. This results in a contradiction, meaning the system has no possible solution.
- A system has infinitely many solutions if at least one free variable exists. A free variable is one that does not have a leading coefficient (pivot) in its column. In such cases, the dependent variables are expressed in terms of the free variables, giving an infinite number of possible solutions.

Vectors

Definition

A vector is a special type of matrix that consists of either a single row (row vector) or a single column (column vector). Vectors are commonly represented using bold letters or sometimes with a bar or arrow above the letter to indicate their vector nature.

Row vector

A row vector is a matrix with only one row. Its general form is:

$$a = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}$$

Column vector

A column vector is a matrix with only one column. Its general form is:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

Vectors without the same dimensions cannot be added. Both vectors must be either both column or row vectors to add or subtract both vectors.

Linear combination

A vector v is a linear combination of vectors a and b if there exists a real number c and d such that v = ca + db

Inner product (dot product)

Let us consider two vectors A and B, with n elements each. Both vectors have the same dimensions:

$$A = \left[a_1\ a_2\ a_3\ \vdots\ a_n\ \right], B = \left[b_1\ b_2\ b_3\ \vdots\ b_n\ \right]$$

The inner product of the vectors A and B is defined as:

$$A.B\ = a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n$$

The product AB can be rewritten as:

$$A.B\ = \sum_{i=1}^{n} a_ib_i$$

The outcome of a dot product is a scalar, not a vector.

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Determinants

Second-order determinants

Second-order determinants are associated with 2×2 matrices. The result of the determinant calculation is a single numerical value.

Let B be a 2×2 matrix. The determinant of B is defined as:

$$|B| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad\ -\ cb$$

Example: Calculate the determinant of the following matrix A.

$$A = \begin{bmatrix} 7 & 2 \\ 6 & 8 \end{bmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 7 & 2 \\ 6 & 8 \end{vmatrix}$$

$$= 7 * 8 - 6 * 2 = 44$$

Determinants of higher order

Expansion by cofactors

$$|B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

A cofactor is determined by taking a line i and a column j of a matrix M and it goes by the following formula:

$$C_{ij} = (-1)^{i+j} \times (\text{the determinant of the matrix without row } i \text{ and column } j \text{ from } B)$$

Example:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 2 & 2 \\ 8 & 1 & 3 \end{bmatrix}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 8 & 3 \end{vmatrix} = -(2*3-8*2) = 10$$

The determinant of the matrix N, mentioned above, can be written in terms of cofactors of either a row or a column.

Cofactor expansion

$$|B| = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

By using cofactor expansion of row i in the matrix B, we get:

$|B| = b_{i1}C_{i1} + b_{i2}C_{i2} + b_{i3}C_{i3}$

Cofactor expansion of column j

By using cofactor expansion of column j in the matrix B, we get:

$|B| = b_{1j}C_{1j} + b_{2j}C_{2j} + b_{3j}C_{3j}$

N-th order determinants

Let us consider the matrix B, with n rows and n columns (nxn).

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

By using cofactor expansion as described before, this time applied to a matrix with more than three rows and three columns, we get:

1. Cofactor expansion of row i:

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \dots + a_{in}C_{in}$$

2. Cofactor expansion of column j:

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \dots + a_{nj}C_{nj}$$

Basic rules of determinants

1. If all elements in a row or a column of matrix **A** are zero, then the determinant is zero: |A|=0.
2. The determinant of a matrix is equal to the determinant of its transpose: (|A|=|A′|).
3. If a matrix C is obtained by multiplying all elements of a single row (or column) of A by a scalar α, then: |C| = α|A|.

4. If two rows (or columns) of a matrix are interchanged, the sign of the determinant changes (multiplied by −1).
5. If two rows (or columns) of a matrix are proportional (linearly dependent), the determinant is zero: |A| = 0.
6. Adding a multiple of one row (or column) to another row (or column) does not change the determinant.
7. The determinant of the product of two square matrices equals the product of their determinants: |AB| = |A| · |B|.
8. If matrix **A** is of size n×n, then multiplying **A** by a scalar α gives: α^{n} |A|
9. In most cases, $|A + B| \neq |A| + |B|$.

Inverse of a matrix

Definition

An n×n matrix B is said to be invertible (or non-singular) if there exists a matrix X such that:

$BX = I_n$ (1) and $XB = I_n$ (2)

In this case, X is called the inverse of B, and it is denoted by B^{-1}.

The inverse of a matrix, if it exists, is unique.

Theorem – Condition for Invertibility

A square matrix B is invertible if and only if its determinant is non-zero:

B is invertible ⇔ det(B)≠0.

Inverse of a 2x2 matrix

Provided that |C| = ad – bc ≠0,

$$C = |a\ b\ c\ d| \rightarrow C^{-1} = \frac{1}{ad-bc} |d\ -b\ -c\ a|$$

Determining the inverse of a matrix

To find the inverse of a square matrix A of size n×n, we use the following procedure:

1. Form the Augmented Matrix:
   Create an augmented matrix by placing the identity matrix In to the right of A, forming a n×2n matrix: [A|In];
2. Apply Elementary Row Operations:
   Use row operations (as in Gaussian elimination) to transform the left side of the augmented matrix into the identity matrix. The goal is to reach: [In|A-1|.

If the row operations successfully convert A into In, then the matrix on the right-hand side will be the inverse of A, denoted A-1.

Important Note: To confirm that the matrix you found is indeed the inverse, it is advised to check that AxA-1=In;

Properties of the Inverse matrix

Let A and B be two invertible n×n matrices, and let ccc be a nonzero scalar. Then the following properties hold:

1. The inverse of A^{-1} is A itself: $(A^{-1})^{-1} = A$
2. The product AB is invertible, and its inverse is given by: $(AB^{-1})^{-1} = B^{-1} A^{-1}$

3. The transpose of an invertible matrix is also invertible, and: $(A')^{-1} = (A^{-1})'$
4. If c≠0, then the inverse of a scalar multiple of a matrix is: $(CA)^{-1} = A^{-1}/C$

Solving a system with the Inverse matrix

This method is only applicable when the system has a **unique solution**, i.e., when the coefficient matrix is **invertible**.

Consider the linear system: Cx = b

- If the matrix C is invertible (i.e., det(C)≠0), we can solve for x by multiplying both sides of the equation by the inverse: $C^{-1} * Cx = C^{-1} * b$
- And finally, x is equal to $x = C^{-1} * b$

Determining the inverse with elementary row operations

When determining the inverse with elementary row operations, simply construct the augmented matrix [C|I] and apply elementary row operations to transform the matrix into [I |B]. Then, $C^{-1} = B$.

Determining the inverse with cofactors

Cofactor expansion provides a key identity for computing the inverse of a matrix. For a square matrix A, cofactor expansion along row iii and using cofactors from row k yields: $a_{i1}C_{K1} + a_{i2}C_{K2} + \ldots + a_{in}C_{Kn} = |a| \text{ if } i = k;\ 0 \text{ if } i \neq k$

This leads to the matrix identity:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$adj(A) = \begin{vmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{vmatrix}$$

This method provides an alternative way to compute the inverse of a square matrix using its adjoint (also called adjugate) and its determinant.
If A is an invertible matrix, then its inverse can be found using the following formula:

$$A^{-1} = \frac{adj(A)}{|A|}$$

Cramer’s rule

Consider a system of n linear equations in n variables, which can be written in matrix form as:

$$Ax=b$$

This system has a unique solution if and only if the coefficient matrix A is invertible, i.e.,

$$\det(A) \neq 0$$

When this condition is satisfied, the solution can be found using Cramer’s Rule:

$$x_1 = \frac{D_1}{|A|}, x_2 = \frac{D_2}{|A|}, \ldots, x_n = \frac{D_n}{|A|}$$

Here, D_j is the determinant of the matrix obtained by replacing the j-th column of A with the column vector b.

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