EFR summary

Mathematics and Game Theory

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Lectures 1 to 3

Weeks 1 to 3

Details

Subject: Mathematics and Game Theory

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Mathematics & Game Theory – Lecture 1, week 1

Game Theory

Definition

Game - a situation where several players take actions that affect the other. A decision made by one player can have an impact on the payoffs of the other players.

Criteria to describe a game mathematically:

- A set of players (e.g., two players in a game of Rock, Paper, Scissors).
- A complete description of the available actions for each player.
- Information structure—what players know at the time of making a decision (e.g., whether they observe each other's moves).
- Outcome function—a rule determining how players' actions lead to different outcomes.
- Players' preferences over outcomes—expressing their ranking or desirability of different possible results (e.g., both players prefer to win).

Basic assumptions:
The game is common knowledge:

- each player knows the game being played,
- each player knows that every other player knows the game being played,
- each player knows that every other player knows that each player knows the game being played, etc.

Rationality:

- Players are assumed to act rationally, meaning they always select the action that maximizes their payoff (utility).

Forms of Games

Mathematical models represent games in two primary forms: **Extensive Form** and **Normal Form**.

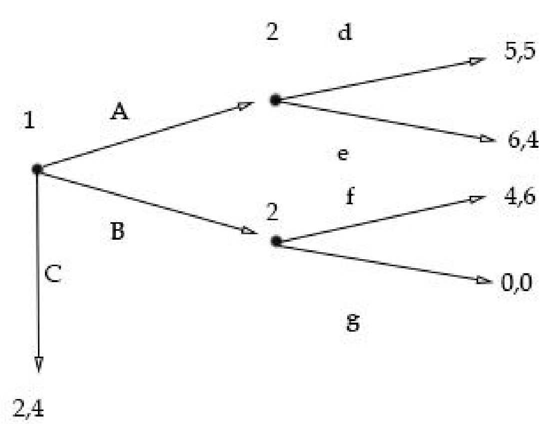
Extensive Form

Definition

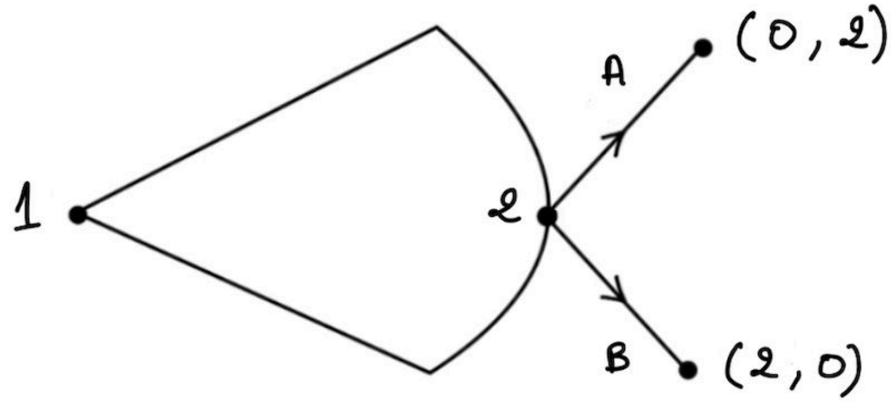
The **extensive form** represents a game as a **tree structure**, detailing the sequence of decisions made by players and the possible payoffs. This format allows for representation of **imperfect information** (where players may not have complete knowledge of past actions).

Information Set:

- Nodes and branches represent decision points and possible actions, respectively.
- Information sets group decision nodes that are indistinguishable to a player (depicted by dashed lines).
- Perfect information games (e.g., Chess) have no dashed lines since all actions are observable.
- Imperfect information games (e.g., Poker) include dashed lines, indicating hidden actions.

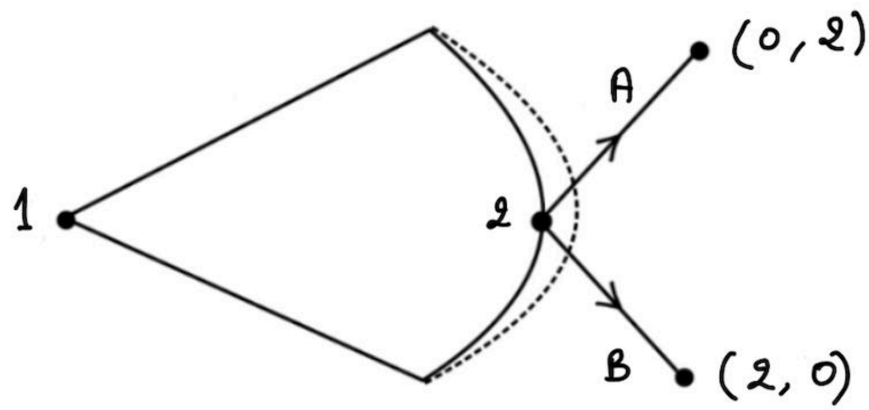


Games featuring an infinite set of actions can be visually represented using an arc connected by two branches:



In the accompanying figure, Player 1 has an unlimited number of possible actions. Consequently, Player 2 faces an infinite number of decision points and must choose between strategy "A" or strategy "B" in response.

In cases of imperfect information—where Player 2 cannot observe Player 1's action—all of Player 2's decision points merge into a single information set. This scenario is depicted using a dashed line, illustrating the uncertainty in Player 2's decision-making process.



Key points:

- There is always one initial node
- Every branch has a label
- Every terminal node has a payoff rector
- At every non-terminal node, indicate the player

Tree rules:

- Tree rule 1: Every node is a successor of the initial node, and the initial node is the only one with this property.

- Tree rule 2: Each node except the initial node has exactly one immediate predecessor. The initial node has no predecessors.
- Tree rule 3: Multiple branches extending from the same node have different action labels.
- Tree rule 4: Each information set contains decision nodes for only one of the players.
- Tree rule 5: All nodes in a given information set must have the same number of immediate successors and they must have the same set of action labels on the branches leading to these successors.
- Tree rule 6 (perfect recall): players remember their own past actions as well as any other events that they have observed.

Normal form

Definition

The **normal form** (also called strategic form) represents a game using a payoff matrix, making it useful for identifying dominant strategies and Nash Equilibria.. The normal-form representation of a game includes all perceptible and conceivable strategies, and their corresponding payoffs, for each player.

Components of Normal Form

In a normal form, the set of players can be depicted inside brackets, with each player separated by a comma. , e.g., {A, B, C} or {1, 2, 3}.
Strategy spaces: Each player has a set of possible strategies, denoted as : $S_1, S_2, S_3, ..., S_n$ for each player in the game.
Payoff functions: The utility function u_i(s) defines player i's payoff for each strategy profile, can be depicted as: $U_1, U_2, U_3, ..., U_n$.

Infinite strategy spaces

In normal-form games with infinite strategy spaces, players are not confined to a limited set of predefined actions or choices. Instead, they can select from an infinite number of possible strategies, typically represented by continuous variables.

Example:

In the mid-1800s, an economist developed a model to describe how two firms compete in a market by deciding how much to produce. Suppose that firms A and B manufacture an identical product, meaning consumers are indifferent to purchasing from either firm.
Each firm selects its production level simultaneously and independently. Let $q_A \geq 0$ represent the quantity produced by Firm A (in thousands of units) and $q_B \geq 0$ represent Firm B's production. The total market supply is then given by:

$$q_A + q_B$$

The price of the product is determined by a simple demand function:
p=1000-(q_A+q_B) Additionally, each firm incurs a production cost of €150 per thousand units produced.

$N = \{A,B\}$

$S_A = S_B = [0, \infty)$

$U_A(q_A, q_B) = 1000 - (q_A + q_B).q_A - 100q_A$

$U_B(q_A, q_B) = 1000 - (q_A + q_B).q_B - 100q_B$

Normal Form Games: Matrix Games

For **two-player** and **three-player** normal-form games with finite strategy spaces, the game can be effectively described using a **payoff matrix**. Each row represents a possible strategy for one player, while each column (or additional dimension for three-player games) represents the strategies of the other player(s). The matrix entries contain the corresponding payoffs for each player based on their chosen strategies.

This matrix representation provides a clear and structured way to analyze strategic interactions, determine best responses, and identify equilibrium solutions such as Nash equilibria.

Example:
Three colleagues are deciding whether to go to a café. Each of them prefers to go only if exactly one other person also decides to go. If only one person goes alone, they will feel awkward, and if all three go, the café will be too crowded and uncomfortable.

The payoffs for each individual are as follows:

- If you stay home, your payoff is 3.

- If you go to the café alone, your payoff is 1.
- If you go to the café with one other colleague, your payoff is 4.
- If all three go together, your payoff is 0.

Draw the normal-form matrix.
Player 3:C

1\2	B	H
B	0,0,0	4,3,4
H	3,4,4	3,3,1

Player 3:H

1\2	B	H
B	4,4,3	1,3,3
H	3,1,3	3,3,3

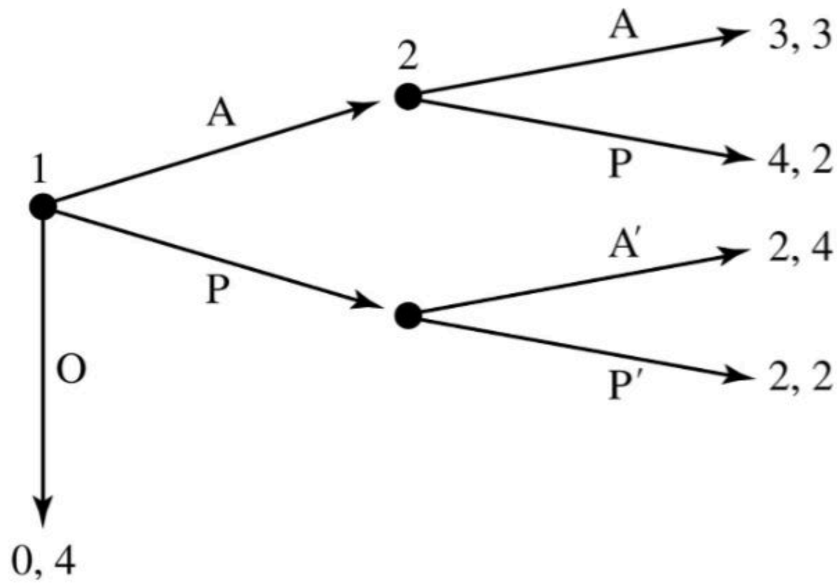
Extensive form vs Normal form

Feature	Extensive Form	Normal Form
Representation	Game tree (nodes & branches)	Payoff matrix
Best suited for	Sequential games, imperfect information	Simultaneous games, clear payoffs
Conversion	Can be converted to normal form in one unique way	Can be represented in multiple extensive forms

- Converting an extensive-form game to normal form results in a single unique representation.
- Converting a normal-form game to extensive form can be done in multiple ways.
- Extensive form is preferred for games involving a strict sequence of actions or imperfect information.

Example:

Draw the normal-form matrix of this game.



1\2	AA'	AP'	PA'	PP'
A	3,3	3,3	4,2	4,2
P	2,4	2,2	2,4	2,2
O	0,4	0,4	0,4	0,4

Mathematics & Game Theory – Lecture 2, week 2

Static Games

Static games refer to strategic interactions where all players make their decisions simultaneously, without knowing what choices the other players have made. Since decisions are made at the same time and independently, players must anticipate their opponents' choices based on available information. These types of games are best represented using normal form models, which display the players, their possible strategies, and the corresponding payoffs in a structured format.

Belief: In a static game, each player forms an expectation about what strategies the other players might choose. This expectation is known as a belief, which is represented mathematically as a probability distribution over the possible strategies of opponents. Essentially, a belief quantifies how likely a player thinks each possible move of their opponents is.

Mixed strategy: occurs when a player does not commit to a single specific move but instead chooses among available strategies according to a probability distribution. This means that rather than always selecting the same action, the player randomizes their choice based on assigned probabilities.

Pure strategy: a mixed strategy that assigns all probability to one strategy. Note: a pure strategy is also a mixed strategy, but not necessarily the other way around.

Expected payoff: the "weighted average" payoff a player gets for a given strategy profile.

Strict Dominance

In game theory, a **strictly dominated strategy** is a strategy that is always worse than another available strategy, regardless of how the opponents play. This means that there exists another strategy—either pure or mixed—that provides a higher payoff in every possible scenario. Because strictly dominated strategies are always suboptimal, rational players will never choose them.

Method for checking whether a strategy is dominated:

- First, decide whether it is dominated by another pure strategy:

see if there is another **pure strategy** that always yields a higher payoff than the one in question, regardless of what the other players do.
If such a better pure strategy exists, then the given strategy is strictly dominated and should never be played.

- Otherwise, check whether it is dominated by a mixed strategy:

1. Look for <u>alternating patterns</u> of large and small numbers in the payoff matrix, which may indicate that a mixed strategy can provide a consistently better outcome.

2. Write down the corresponding equations.
3. Solve these equations to determine if a mixed strategy dominates the given strategy.

1\2	A	B
M	3,4	7,0
S	2,0	5,4

Determine which strategies are dominated:
Player 1: S is dominated by M
Player 2: No dominated strategy

Iterated Dominance

- In strategic decision-making, a **dominated strategy** is one that always results in a worse outcome than another available strategy, regardless of what the opponents choose. Since rational players aim to maximize their payoffs, they will never play a dominated strategy.
- **Iterated dominance**: repeatedly eliminating dominated strategies from the game until only the most rational choices remain.
    - Delete all of the dominated strategies for each player.
    - R1 are the strategy profiles that remain.
    - Delete from R1 any strategies that are dominated in this reduced game.
    - R2 are the strategy profiles that remain.

Note: The order in which strategies are eliminated **does not matter** as the final result will always be the same.

Rationalizable strategies: the set of strategies that survive iterated dominance, By applying iterated dominance, players can simplify strategic decision-making and focus only on rational choices that maximize their expected payoffs.

Example:

a. Determine the set of rationalizable strategies in the following game.

1\2	A	B	C
D	3,7	0,5	5,5
E	4,4	0,0	2,6
F	2,2	4,6	3,4

Solution:

the rationalizable strategies in the game :
Player 1: {D,E,F}
Player 2: {A,B,C}

Best Response

A **best response** is the strategy that provides the highest possible payoff for a player, given the strategies chosen by the other players. In other words, it is the optimal decision a player can make when assuming that their opponents' strategies are fixed.

Example:

a. *Determine* $BR_{1(\frac{1}{3}, \frac{1}{2}, \frac{1}{4})}$

1\2	E	F	G
A	3,7	0,5	5,5
B	4,4	0,0	2,6
C	2,2	4,6	3,4

Solution:

- If player 1 selects A, his payoff would be: $\frac{1}{3} * 3 + \frac{1}{2} * 0 + \frac{1}{4} * 5 = 2.25$
- If player 1 Selects B, his payoff would be: $\frac{1}{3} * 4 + \frac{1}{2} * 0 + \frac{1}{4} * 2 = 1.83$
- If player 1 Selects C, his payoff would be: $\frac{1}{3} * 2 + \frac{1}{2} * 4 + \frac{1}{4} * 3 = 3.42$

Hence, the best response for player 2 would be to select strategy A.

Nash Equilibria

Nash equilibrium: strategic outcome in which no player can improve their payoff by unilaterally changing their strategy, assuming that the other players keep their strategies unchanged. In other words, at a Nash equilibrium, every player is playing their **best response** to the strategies chosen by the others.

Nash equilibria can be:

- Pure-strategy (by removing rows and/or columns that are dominated)
- Mixed-strategy:
  - involves players randomizing their choices based on a probability distribution.
  - Set up equations that represent the expected payoffs for different strategies..
  - Solve these equations to determine the probabilities with which players should mix their strategies.
    - Every game with a finite number of players and a finite strategy space has at least one Nash equilibrium.

Compute the pure-strategy Nash equilibria:

1\2	M	N	P
K	6,7	4,8	0,5
L	9,4	4,2	6,3
J	8,6	5,5	6,7
S	4,5	8,6	4,4

3 Pure Nash Equilibria: (L,M), (J,P), (S,N)

Dynamic Games

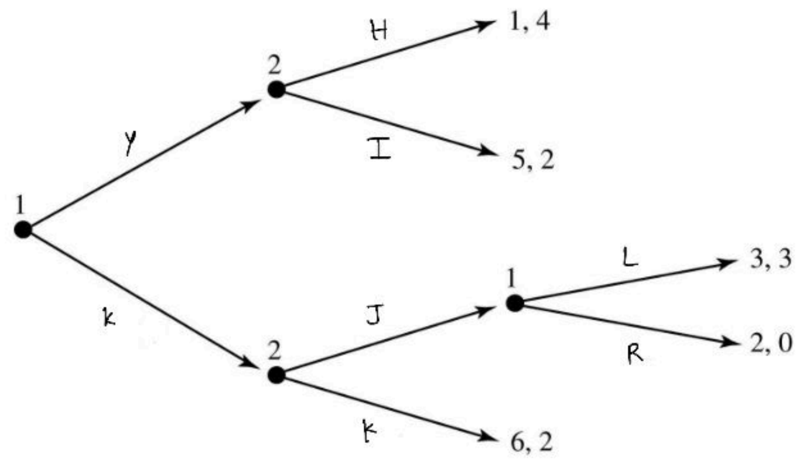
Backward induction

In **dynamic games**, players make decisions sequentially according to a predetermined timeline. These games are best represented using the extensive form, which captures the order of moves and possible choices at each stage.

Backward induction: a method used to solve such games by working in reverse, starting from the final decision and moving backward to the beginning. At each step, suboptimal choices are eliminated, ensuring that only rational decisions remain.

This approach is only applicable to games with **perfect information**, where each player is fully aware of all previous moves when making a decision.

Solve the game by using backward induction:



(KL, HJ)

Subgames

A **subgame** is a distinct part of an extensive-form game that satisfies the following conditions:

- It begins at a single **initial node**, which must be the only node in its information set.
- If a node is included in the subgame, all of its **successor nodes** must also be included.
- If any node from a particular **information set** is in the subgame, then all nodes in that information set must be part of the subgame as well.

Types of Subgames

Proper subgame: a subgame that starts from nodes other than the initial node.
Minimal proper subgame: a proper subgame that does not contain another proper subgame.

Subgame perfect Nash equilibria

Subgame perfect Nash equilibrium (SPNE): a strategy profile in which players play a **Nash equilibrium** in every subgame of the original game. This ensures that players make optimal choices at every stage, not just in the overall game but also within any possible subgame.

- SPNE can be both pure-strategy and mixed-strategy.
- Finding SPNE:
    1. Identify the minimal proper subgame and determine a Nash equilibrium within it.
    2. Replace this subgame with the payoff vector corresponding to the selected equilibrium.
    3. Repeat the process for the reduced game until reaching the initial decision node.
    - SPNE is most effectively applied in perfect information games using backward induction, ensuring rational decision-making from the last move to the first.

Mathematics & Game Theory - IBEB - Lecture 3, week 3

Integration

Definition

Integration is the reverse process of differentiation, often called taking the antiderivative. It involves determining the original function given its derivative, essentially reconstructing the function from its rate of change.

Types of Integrals

There are two types of integrals, namely definite and indefinite integrals.

Indefinite Integrals

Definition

The function F(x) is called an indefinite integral of f(x) if f(x)=F′(x). In other words, integration is the reverse process of differentiation.

- Indefinite integrals do not have specified limits and are expressed as:

$$\int f(x)dx = F(x) + C$$

- where C is an arbitrary constant. This constant arises because differentiation of $F(x) + C$ still results in f(x), since the derivative of a constant is zero.
- Indefinite integrals are also known as antiderivatives.

Applications for integrals

1. Computing areas in graphs
   a. Integration is used to calculate the area under a curve, which is particularly useful in geometry, physics, and economics.
2. Computing probabilities
   a. In probability theory, integration is used to find the probability of an event occurring within a specific range on a probability density function (PDF) curve.
3. Consumer and Producer Surplus
   a. When demand and supply functions are not linear, integration helps determine consumer and producer surplus by computing the exact area between the curves.

Important Integrals

Here is a list of important integrals to be used in this course:

- $\int x^b dx = \frac{1}{b+1}x^{b+1} + C$, where b≠-1
- $\int e^{bx} dx = \frac{e^{bx}}{b} + C$ provided that b≠0

- $\int \frac{1}{x} dx = ln|x| + C$
- $\int a^x dx = \frac{a^x}{lna} + C$, where a>0 and a≠1.

General rules for integrals

Let b≠0 be a constant.

- $\int bf(x)dx = b\int f(x)dx$
- $\int (f(x)\pm g(x))dx = \int f(x)dx \pm b\int g(x)dx$

Example: $\int \frac{1}{2}x + 5e^{x} dx = \frac{1}{2}\int x\ dx + 5\int e^{x} dx$

Definite Integrals

Definition

A **definite integral** is an integral with specified lower and upper limits, which define the interval over which the function is integrated. Given a function f(x) over the interval [a,b], the definite integral is written as: $\int_a^b f(x)dx$.

Unlike indefinite integrals, definite integrals evaluate to a numerical value rather than a function. If F(x) is an antiderivative of f(x), meaning F'(x)=f(x), then the definite integral is calculated using the **Fundamental Theorem of Calculus**:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where F(x) is any indefinite integral of f(x).

How to Evaluate a Definite Integral:

1) Find the indefinite integral F(x) of f(x).
2) Evaluate F(x) at the upper limit b.
3) Evaluate F(x) at the lower limit a.
4) Subtract: F(b)-F(a).

Properties

Let $f(x)$ be a continuous function on an interval containing c,d,e,m and n, and let βbe a constant. The following properties hold for definite integrals:

1. $\int_m^n f(x)dx = -\int_n^m f(x)dx$
2. $\int_c^c f(x)dx = 0$
3. $\int_m^n \beta f(x)dx = \beta\int_m^n f(x)dx$
4. $\int_m^n [f(x) + q(x)]dx = \int_m^n f(x)dx + \int_m^n q(x)dx$
5. $\int_c^d f(x)dx = \int_c^e f(x)dx + \int_e^d f(x)dx$

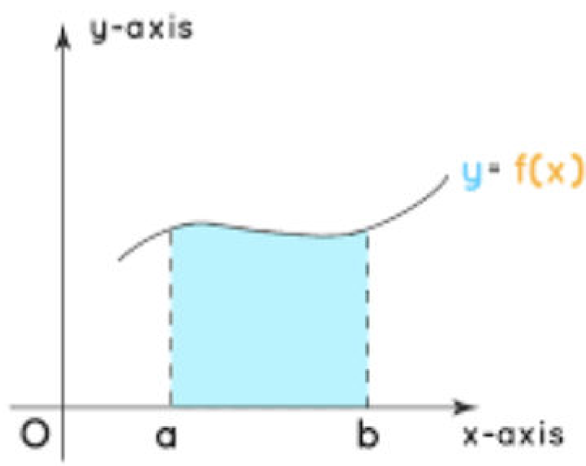
Example: Compute the integral $\int_0^2 x^2 + 5\,dx$

$$= \left[\frac{x^3}{3} + 5x\right]_0^2$$

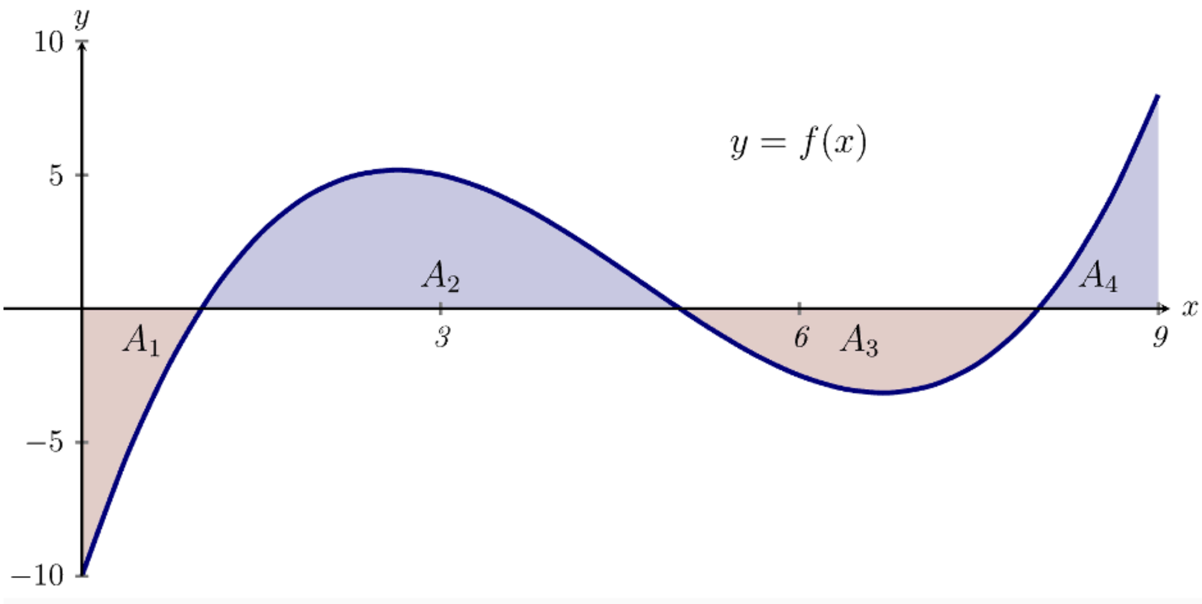
$$= \left(\frac{2^3}{3} + 5 * 2\right) - \frac{0^3}{3} + 5 * 0 = \frac{8}{3} + 10$$

Areas and Definite Integrals

Definite integrals can be used to compute the area between functions or a function and a line. As in this case:



The area represented by green can be computed by using the formula $\int_a^b f(x)dx$, if f(x)≥0 for every x and $-\int_a^b f(x)dx$ if f(x)≤0 for every x.



Derivative of an Integral

Let us denote $A(x) = \int_{f(x)}^{q(x)} h(t)dt$. After solving the definite integral, we will eventually reach $A(x) = B(t)_{f(x)}^{q(x)} = B(q(x)) - B(f(x))$, where B'(t)=b(t). The integral of this function will be written in terms of x, therefore we can also find its derivative. Thus, A'(x) is given by the following formula:

$$\frac{d}{dx}A(x) = \frac{d}{dx}\int_{f(x)}^{q(x)} b(t)dt = \frac{d}{dx}(B(q(x)) - B(f(x))) = b(q(x))g^{'}(x) - b(f(x))f'(x)$$

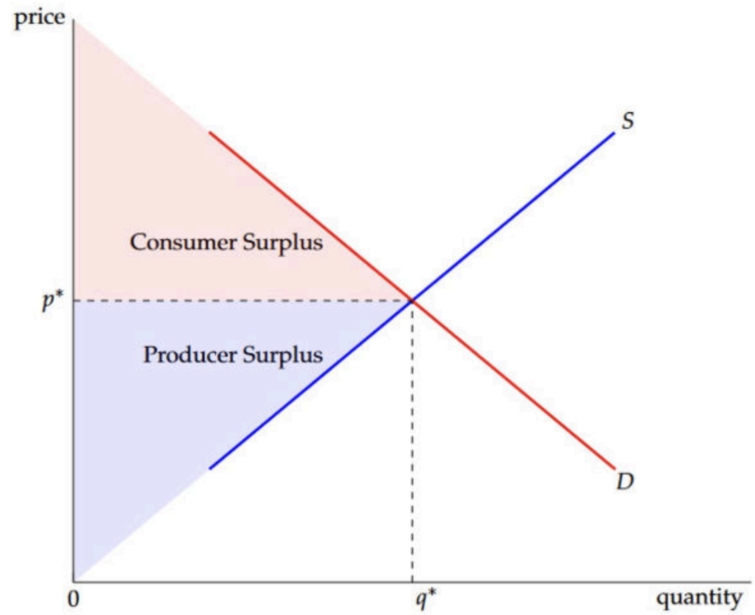
In short, $A^{'}(x) = h(g(x))g^{'}(x) - h(f(x))f'(x)$.

Consumer and Producer Surplus

The consumer and producer surplus can be determined by identifying the intersection of the demand curve and the supply function along with the Oy-axis.

To calculate these areas using definite integrals, we first need to find the equilibrium point (P*,Q*), where the demand and supply functions intersect. Once this point is identified, we can apply the formula for finding the area between two curves.

Let D(Q) represent the demand function and S(Q) represent the supply function. The consumer surplus is the area between the demand curve and the equilibrium price, while the producer surplus is the area between the equilibrium price and the supply curve. These values can be determined using integration.



Consumer Surplus

We have the following formula for determining consumer surplus (CS):

CS= $\int_{0}^{Q*} (D(Q) - P*)dQ$

Producer Surplus

We have the following formula for determining producer surplus (PS):

$$PS=\int_{0}^{Q*} (P* - S(Q))dQ$$

The Substitution Rule

Indefinite Integrals

The substitution rule is a powerful technique in integral calculus that simplifies the process of evaluating indefinite integrals. It states that if we define u=g(x), then the derivative g'(x)dx can be expressed as du. This substitution transforms the given integral into a more manageable form while preserving the original integral's solution. Mathematically, the substitution rule is expressed as:

$$\int f(g(x))g^{'}(x)dx = \int f(u)du = F(u)$$

where F'(u)=$f(u)$

Once the indefinite integral F(u) is determined, we substitute u back with g(x), yielding the final result:

$$\int f(g(x))g^{'}(x)dx = F(g(x))$$

Definite Integrals

The substitution rule for definite integrals works similarly to that for indefinite integrals, with the key difference that the limits of integration must also be adjusted based on the substitution.

$$\int_{m}^{n} f(g(x))g^{'}(x)dx = \int_{g(m)}^{g(n)} f(u)du$$

Key Steps for Applying the Substitution Rule in Definite Integrals

1. Choose a substitution: Let u=g(x), then compute du=g'(x)dx.
2. Adjust the limits of integration: Transform the original limits x=m into u-values by substituting them into u=g(x).
3. Rewrite the integral in terms of u: Replace all occurrences of x with u, including the differential dx.
4. Evaluate the new integral: Compute the definite integral in terms of u and evaluate it using the transformed limits.

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