EFR summary

Introduction to Mathematics, FEB11003X

2023-2024

Lectures 1 to 7

Weeks 1 to 7

Details

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Introduction to mathematics – IBEB – Lecture 1 & 2 – week 1 & 2
Basic algebra and equations

Logic(Implication Arrows)

A=>B
"A implies B"
"A is a sufficient condition for B"
"B is a necessary condition for A"

It is important to remember that even if A implies B, this does not always mean that B implies A.
If A implies B, AND B implies A, then we can use "⇔"
"⇔" is called the equivalency arrow. This arrow can be used in "if and only if" statements.

Powers

Table of power rules (+, −, ×, ÷)

Name of the rule	Rule	Example
Product	$x^a \cdot x^b = x^{a+b}$	$2^2 \cdot 2^3 = 2^{2+3} = 32$
	$x^a \cdot y^a = (x \cdot y)^a$	$2^2 \cdot 3^2 = (2 \cdot 3)^2 = 36$
Quotient	$x^a / x^b = x^{a-b}$	$2^6 / 2^1 = 2^{6-1} = 32$
	$x^a / y^a = (x / y)^a$	$4^4 / 2^4 = (4/2)^4 = 16$
Power	$(y^a)^b = y^{a.b}$	$(2^4)^3 = 2^{4.3} = 4096$
	$\sqrt[a]{(y^b)} = y^{b/a}$	$\sqrt[3]{(2^9)} = 2^{9/3} = 8$

	$y^{1/2} = \sqrt{y}$	$16^{1/2} = \sqrt{16} = 4$
	$y^{1/a} = \sqrt[a]{y}$	$16^{1/4} = \sqrt[4]{16} = 2$
Negative index (exponent)	$y^{-a} = 1 / y^{a}$	$2^{-2} = 1/2^{2} = 0.25$
"0" rules	$y^{0} = 1$	$3^{0} = 1$
	$0^{a} = 0$, for $a > 0$	$0^{6} = 0$
	0^{a} is not defined if $a \leq 0$	(this would result in dividing by 0)
"1" rules	$y^{1} = y$	$10^{1} = 10$
	$1^{n} = 1$	$1^{542} = 1$

All of the basic algebra rules can be applied in any kind of equation as long as the rules are respected (e.g. for fractional powers we can apply both the rules for fractions and for powers).

Fractional powers

A fractional exponent or power is an alternate notation for expressing powers and roots together (Betty, Brat. Medium.com, 2015) For example,

$$a^{\frac{1}{2}} = \sqrt{a}$$
$$a^{\frac{1}{b}} = \sqrt[b]{a}$$

where we could see how we write the power in the numerator and the index of the root in the denominator.

Calculations with square roots

1. $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$
2. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Where $b \neq 0$

Note: $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$

Fractions

A fraction is a part of a whole. Its main characteristics are the **numerator**, the number or function that can be found above the fraction line, and the **denominator,** the number or function that is under the fraction line.

Recall that:s

$a \div b = \frac{a}{b}$

Where *a* is the **numerator** and *b* is the **denominator**.

Properties of fractions (+, −, ×, ÷)

1. $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$
2. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$
3. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$
4. $\frac{1}{\frac{1}{a}} = a$
5. $\frac{a}{a+b} \neq \frac{a}{a} + \frac{a}{b}$, where a, b, a+b ≠ 0

Useful tip:

When solving fractions:

- Simplify the fractions as much as possible <u>before</u> trying to calculate them (for example, by using the properties of fraction to make it as simple as possible)
- Always try to bring the fractions to a common denominator by factorizing

Simplification/amplification

Let x, y, z be three real numbers, with b, c $\neq 0$.

$\frac{ac}{bc} = \frac{a}{b} \times \frac{c}{c} = \frac{a}{b}$ is an example of **simplifying fractions.**

$\frac{a}{b} = \frac{a}{b} \times \frac{c}{c} = \frac{ac}{bc}$ is an example of **amplifying fractions.**

Inequalities

If "*a*" is a positive number, we write $a > 0$ (or $0 < a$), and we say that *a* is greater than zero. If the number *c* is negative, we write $c < 0$ (or $0 > c$).

- The principles for solving inequalities are quite similar to the rules for solving linear equations, and they are easy to remember. In the case of negative numbers, there is one exception: when multiplying or dividing by a negative number
  - When multiplying inequalities, do not forget to flip the sign if you multiply or divide the equation with a negative.

Properties of inequalities

Let *a*, *b*, *c*, and *d* be numbers:

1. $(a>b \text{ and } b>c) \Rightarrow a>c$
2. $(a>b \text{ and } c>0) \Rightarrow ac>bc$
3. $(a>b \text{ and } c<0) \Rightarrow ac < bc$
4. $(a>b \text{ and } c>d) \Rightarrow a+c>b+d$

The sign table

The **sign table** is used to determine the sign of a function when knowing the simple factors that compose it through different operations (like multiplication, division). This could be used to solve inequalities directly or when asked to determine the values of x for an expression when it is either negative, positive or zero.

Example 1: Find the values of x for which $(x-2)(x-7)(x^2-9)>0$.

First, we decompose the left side of the inequation into simpler terms. We have: $x^2-9 = x^2-3^2 = (x-3)(x+3)$. *This leads to the simplest decomposed form of the inequation:* $(x-2)(x-7)(x-3)(x+3)>0$.

In order to create the sign table, we need to determine the sign of each of the brackets. Therefore, x-2 is positive when x>2, negative when x<2 and 0 when x=2. We apply the same rule for the rest of the brackets.

There are much more complicated functions in brackets that necessitate more operations in order to determine their sign (like differentiation). And so, let us create the sign table for the expression (x-2)(x-7)(x-3)(x+3), to determine for which values of x it is positive.

x	-∞	-3	2	3	7	+∞
x+3	---------- ----	-------0 +++	+++++++	++++++	++++++++	++++++
x-2	---------- ----	--------- ----	------0+ ++	++++++	++++++++	++++++
x-3	---------- ----	--------- -----	--------- ----	------0+ ++	++++++++	++++++
x-7	---------- ----	--------- -----	--------- -----	--------- -----	-------0+ ++	++++++
Expression	++++++++	++++0-- ---	------0+ ++	+++0---- ---	-------0+ ++	++++++

From the sign table we notice that the expression is positive for x $\in$ *(-∞,-3)* $\cup(2,3)\cup(7,\infty)$*, zero for x = -3, 2, 3, 7 and negative for the rest of the values.*

Useful tip:
As shown on the example above, before using the sign table to determine the sign of the function, we can break down the equation into different parts in which they are multiplied by.

Example:

Determine the values of x for which the expression is positive, zero and negative.

$-(x-1)^2\frac{x^2-2}{2e^x-4}$

Break down of expression into several parts:

1. $-(x-1)^2$
2. x^2-2
3. $\frac{1}{2e^x-4}$

And then continue on determining the sign for the values of x in each separated function using the sign table as shown in example 1.

Intervals

(a,b) - Open interval from a to b

- A < x < b

[a,b] - Closed interval from a to b

- A ≤ x ≤ b

(a,b] - Half-open interval from a to b

- a < x ≤ b

[a,b) - Half-open interval from a to b

- a ≤ x < b

A∈B - A is a member of set B, ie. A belongs to set B
x ≥ 1 : we can also write as x ∈ [1,∞), in which "∞" means **infinity**.

Absolute values

The absolute value of a number defines how far away it is from zero on the number line, without taking into account its orientation.
|b| = {b if b is greater than or equal to 0; -b if b is less than 0}
"b" is called the "argument"

solving linear absolute value equations

Step 1: The absolute value must be isolated
Step 2: Identify: What is the isolated absolute value equal to?

- If absolute value = 0: Remove the absolute value symbols from the equation and solve it to obtain a **single result**.
- If absolute value < 0: **no solution**
- If absolute value > 0: Insert an 'or' statement in between the two equations and may make the "argument" equal to both the number and the number's opposite. Then, for each equation, solve it independently to obtain two

possible answers. Then, for each equation, solve it independently to obtain two possible answers.

solving linear absolute value inequalities

Step 1: The absolute value must be isolated
Step 2: Identify: What is the isolated absolute value equal to? If…

"Negative"	"Positive"
• The answer consists entirely of real numbers if the absolute value is higher than, or more than or equal to, a negative number. Anything with a positive absolute value will always be greater than something with a negative absolute value.	• There are two methods to tackle the problem if the absolute value is less than or equal to a positive number, depending on the situation. In either case, the answer will be expressed as an **intersection**. o The argument should be placed in a three-part inequality (compound) between the opposite of the number and the number, then solve o Set the argument so it is smaller than the number and bigger than the number's opposite. Do not forget to insert an "and" statement between the two inequalities. • If the absolute value is higher than or equal to a positive number, use a 'or' expression to set the argument less than the opposite of the number and larger than the number. Then, for each inequality, solve it by expressing the solution as a union of the two.

"Zero"
• If absolute value < 0: **no solution** • Whenever the absolute value is less than or equal to zero, there is only **one possible solution**. Simply set the parameter to zero and solve the problem. • As long as the absolute value is higher than or equal to zero, the solution is comprised entirely of **real numbers.** • As long as the absolute value is larger than zero, the solution is composed of **all real numbers**, with the **exception of the value that brings it equal to zero**. This will be written in the form of a union.

Step 3: Graph the resulting numbers on a number line and write the answer in interval notation.

Summations

The sum, from $i = 1$ to $i = n$, of x_i is,

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

Functions

A function with domain A is a rule that each real variable x in A assigns a single real number f(x) (Sydsæter et al., 2021, p.101).

Follow the form y = f(x)

"x"

- Independent variable
- Also called argument
- Domain: All possible values for x

"y"

- Dependent variable
- Range: All possible values for y

The simple form of a function

A function is usually composed of three things:

- **domain**: the *input* set of numbers
- **range**: the *output* set of numbers
- The *relationship* that defines the function

A function is usually defined as follows:
Let f be a function, f: A → B, f(x)= *form of the function*

Consider the function f(x) = y :

The values of x that we allow for f constitute the domain of f .

The values y such that y = f (x) for at least one x in the domain of f , constitute the range of f .

Example:

f: R → R, f(x)= $\sqrt[5]{(x+1)}$, where R is the set of real numbers. Compute f(0) and f(31).
By replacing x with 0 and 31 respectively we obtain: f(0)= $\sqrt[5]{(0+1)}$= $\sqrt[5]{1}$= 1 and f(31)= $\sqrt[5]{(31+1)}$= $\sqrt[5]{32}$= 2.

Equation of a line

The equation of a line is represented by y = mx + c, whereas m is the gradient, or slope, (a>0 means the line is increasing and a<0 means the line is decreasing) and b is the distance to the origin. This could be seen in many economic models, such as the demand curve (D=a-bP) and the supply curve (S=a+βP).

Linear functions

Follow the form y = f(x) = ax + b

"a"

- Slope
- If a>0, f(x) is an increasing function
- If a<0, f(x) is a decrease function

"b"

- Distance to the origin
- Measures along the y-axis
- Root:
  ax + b = 0 ⇒ x = - b/a

Polynomials

General form

A **polynomial** of degree n is a function of the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ **=** $\sum_{i=0}^{n} a_ix^i$, with a_i being parameters, where *i* takes all the values from 0 through n.
Consider the integer n, numbers a0, a1, ... , an, with an ≠ 0, and the variable x. A polynomial of degree n is a function of the form:

$$f(x) = a0 + a1x + a2x2 + a3x3 + ... + anxn$$

General form of function (or called "cubic functions"):

$$f(x) = ax^3 + bx^2 + cx + d$$

Example: $f(x) = 1 + 10x + 25x^2 + x^3$ *is a third-degree polynomial.*

Special polynomials

1. **Linear Polynomial**: the highest degree of x is 1. Its form is f(x)=ax + b, with a, b being real numbers.
2. **Quadratic Polynomial**: the highest degree of x is 2. Its form is f(x)= ax^2+bx+c. with a, b, c being real numbers.

Polynomial division

When dividing polynomials, it is possible that we will need to factor the polynomials in order to discover a common factor between the numerator and denominator.

For example: $\frac{(x^2-25)}{(x-5)} = \frac{(x-5)(x+5)}{(x-5)} = \frac{(x+5)}{1}$

When we cannot discover the factors of a number in a short division procedure, we can utilize a longer division method.

- To get the quotient, divide the dividend by the divisor, then multiply the quotient by the divisor and subtract.
- To get the next term of the quotient, just divide the first term of the remaining dividend by the first term of the divisor, which yields the next term of the quotient.

Polynomial division

Polynomials can be divided using the polynomial division rule. When dividing one polynomial with another, we stop the division when the remainder's degree is smaller than the divider's. The polynomial that is being divided is called **dividend** and the one that divides is called **dispenser**. The polynomial that is left and can no longer be divided is called **remainder**.

Example: $(x^4+3x^3+10x+3) \div (x^2+1) = x^2+3x-1$

$-x^4-x^2$

/ $3x^3-x^2+10x+3$

$-3x^3 - 3x$

/ $-x^2+7x+3$

$x^2 + 1$

/ $7x+4$ *remainder*

Hence, $(x^4+3x^3+10x+3) \div (x^2+1) = x^2+3x-1- -\frac{7x+4}{x^2+1}$

Example 2: $(2x^3 + 2x - 1)\div(x - 1)$

$(x - 1)\,/(2x^3 + 2x - 1)\backslash\, 2x^2 + 2x + 4$

$2x^3 - 2x^2$

$2x^2 + 2x - 1$

$2x^2 - 2x$

$4x - 1$

$4x - 4$

3. *remainder*

Hence, $(2x^3 + 2x - 1)\div(x - 1) = 2x^2 + 2x + 4 - \frac{3}{x-1}$

Equations

Equations can appear in all sorts of forms and with all sorts of variables (e.g. fraction equations, power equations, logarithmic equations).

Special equations

Special equations are, as mentioned above, for example the linear and quadratic equations. The quadratic equation's basic form is $ax^2 + bx + c = 0$. To solve it we need to calculate the discriminant denoted with **$\Delta = b^2 - 4ac$**. The solutions of the equation are $x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a}$ and the decomposed form of the equation is $a(x - x_1)(x - x_2)$.

Quadratic equation

Quadratic Equation Form: $f(x) = ax^2 + bx + c$

Quadratic Formula: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Where both "a" and "b^2-4ac" are not equal to 0.

Quadratic functions have an exponent with the highest degree of 2. Graphs of quadratic functions are parabolas.

To solve for “x” in quadratic functions, use:

- Factorization
- Quadratic Formula

discriminant of a quadratic equation

$$\boldsymbol{\Delta = b^2 - 4ac}$$

When Δ>0 □ two distinct real roots (two real solutions)

When Δ=0 □ one real root (one real solution)

When Δ<0 □ no real roots (no real solutions)

Exponential functions

$$F(x) = a^x,\ a>0\ \square\ f>0$$

What is “e”?

“e” is Eulers number (named after Leonhard Euler). It is approximately 2.71828. In mathematics, "e" is the base rate of growth that is shared by all continuously developing processes.

$$\text{Ln}(1) = 0 \Leftrightarrow f(x) = \text{In}(x), \text{ only if } x>0$$

$$\text{Ln}(e) = 1\ \&\ \text{In}(1) = 0$$

Rules for logarithms

For all x > 0 and y > 0,

In(xy) = In(x) + In(y)

In($\frac{x}{y}$) = In(x) - In(y)

In($\frac{1}{x}$) = -In(x)

In(x^n) = n In(x)

Introduction to mathematics – IBEB – Lecture 3 – week 3

Inverses and derivatives

Shifting of graphs

The graph of a function can be shifted on the axes "x" and "y" by adding or subtracting a parameter to the argument and to the value of the function accordingly. Shifting the graph of a function on "y" can be done by simply adding a parameter (with the sign + or -) to the defined value of the function, in which the rules as defined is:

Shifting rules when c,d > 0:

f(x) + c	shifts the function upwards by c
f(x) – c	shifts the function downwards by c
f(x+d)	shifts the function to the left by d
f(x-d)	shifts the function tot he right by d

Example: Given a function f(x)=3x+2, Shift the equation to the right by 3 units and upwards by 5 units

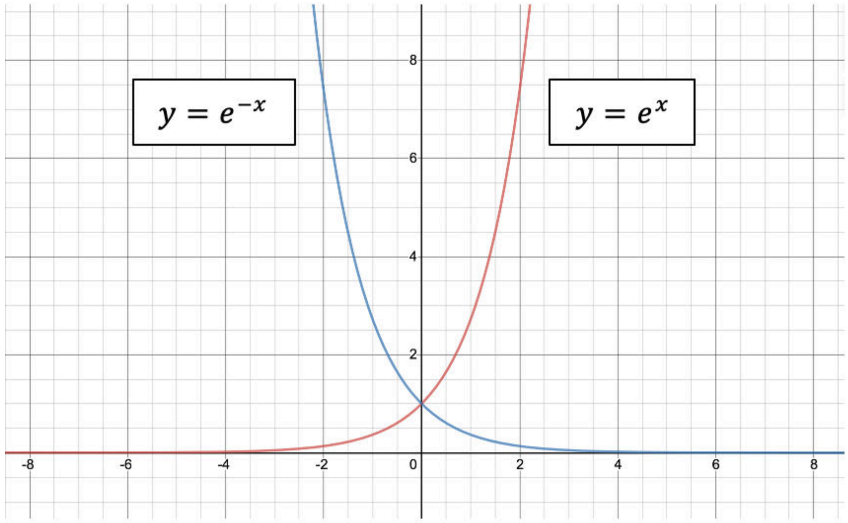
$$f(x) = 3(x - 3) + 2 + 5$$
$$\Rightarrow f(x) = 3x - 9 + 2 + 5$$
$$\therefore f(x) = 3x - 2$$

Stretching and reflecting graphs

<u>Reflection on the *y-axis*</u>

$$f(x) \rightarrow f(-x)$$

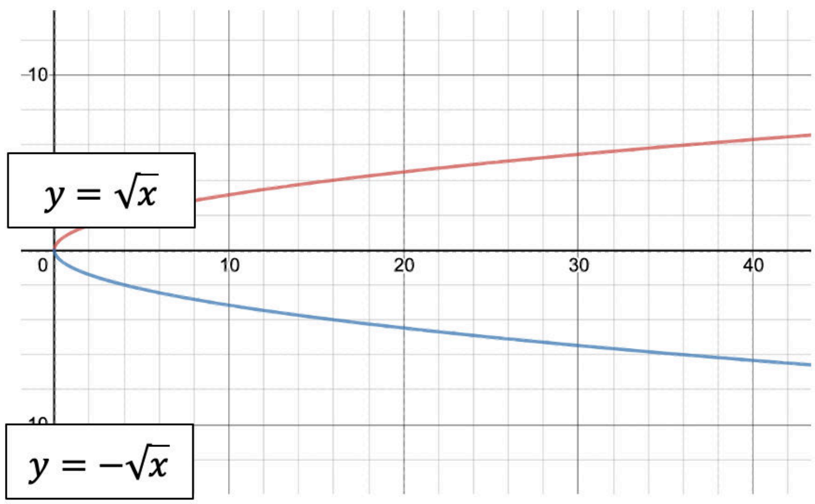
Example:



<u>Reflection on the y-axis</u>

$$f(x) \rightarrow -f(x)$$

Example:



Composite functions

Given two functions *f(x)* and *g(v)*, the composition of *g* with *f* is the function *h(x) = g(f(x))*.

The composition of *g* with *f* is written as “ $g \circ f$”.

Inverse functions

An inverse function (or anti-function) is a function that "reverses" another function: if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, and vice versa.
Graphically, the inverse will be the reflection of the original function in the line y=x

Consider the function F(x) = y,
If for every x in the domain off, and for every y in the range of f it holds that
g(y) =x ⇔ y=f(x), then we call g the inverse of f.

So, x=g(x) and y=f(g(y))

Inverse of f is denoted by f^{-1} and the domain of f is the range of f^{-1} and vice versa.

Note: exists only for the "one-to-one" functions: for each y in the range of f, there exists only one x such that y=f(x) where f is strictly increasing or decreasing.

Example:

$$f(x) = 3x + 2$$

$$y = 3x + 2$$

(move the variables to the left-hand side with the letter y while leaving the x-variable on the other side)

$$y - 2 = 3x$$

$$\frac{y-2}{3} = x$$

$Hence,\ f^{-1}(x) = \frac{x-2}{3}$ ***or*** $f^{-1}(y) = \frac{y-2}{3}$ *(doesn't matter what variable you use)*

Useful tip: As the domain of f is the range of f^{-1} and vice versa, when solving problems that asks about the range or domain of a certain inverse function, it is not necessary to find the inverse of the function first, yet, you can answer directly by finding the domain or range of the actual function given.

> *Example:*
> *Find the domain of the inverse function* $f(x) = \frac{3x+2}{x-2}$
> *As you can see, the domain of the inverse function is the range of the function f(x). Hence, we can note that the range of function f(x) is 3, thus the domain of* $f^{-1}(x)$ *is 3.*

How to find the inverse functions

1. Replace f (x) with y.
2. Replace all "x" to "y" & replace all "y" to "x"
3. Solve the equation for "y"
4. Replace "y" to $f^{-1}(x)$. This is the inverse function!

Introduction to limits

Let f(x) be a function defined on an interval that contains a parameter q. The limit of f(x) is denoted as:

$$\lim_{x \to a} f(x)$$

It shows the value that f(x) approaches when x is chosen arbitrarily close, but not equal, to a.

Example:

$$\lim_{x \to 0} f(x) = \frac{e^x - 1}{x}$$

It could be seen that the function is not defined when x=0, thus, $x \neq 0$ *and 0 is not on its domain. Therefore, we can conclude that the limit is very close to 0.*

What we can do is to check numbers, what numbers does it get very close to, in this case, it would get closer to 1.

$$\therefore \lim_{x \to 0} f(x) = \frac{e^x - 1}{x} = 1$$

Possible ways to find limits

There are different ways to try to find a limit. If one method leads to an undefined number, try another way. These are some methods of find the limit:

1. Plug-in or substitute the value of the limit
2. Factor the equation and see if some terms cancel out

3. Get a common denominator (if the question is a complex fraction)
4. Expand the equation and simplify
5. If the equation has a square root, multiply it with it's conjugate and rationalize the equation.

Rules for limits

1. $\lim_{x \to a} c = c$

2. $\lim_{x \to a} x = a$

3. $\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

5. $\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \lim_{x \to a} g(x) \neq 0$

7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n, \quad \text{where } n \in \mathbb{N}$

8. $\lim_{x \to a} x^n = a^n$

9. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}, \quad \lim_{x \to a} f(x) > 0 \text{ if } n \text{ is even.}$

Derivative

Derivatives are associated with rates of change (e.g. in Economics: growth rate, marginal costs, *etc).* It is also associated with optimization (e.g. in Economics: when trying to maximise profits, minimize costs, *etc)*

The Newton Quotient (also called difference quotient) is defined as:

$$\frac{f(a+h)-f(a)}{h}$$

The derivative of function f(x) at x=a is defined as the limit of the Newton quotient when h tends to 0.

$$\lim_{x \to 0} \frac{f(a+h)-f(a)}{h}$$

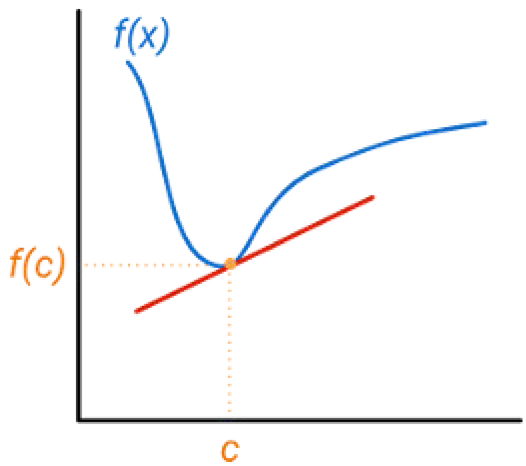
The derivative of f(x) is itself a function of x and is denoted by f′(x)

$$f'(x) = \lim_{x \to 0} \frac{f(a+h) - f(a)}{h}$$

To find the difference quotient at a point, we simply have to plug in the "x" value of the point to the "a" value in Newton's Quotient.

Tangent and notation

A tangent is a line that comes into contact with the curve of a function's graph in only one point. A function can have multiple tangents, just as there are functions that have no tangents to their graphs (e.g the linear function ax+b). The tangent line At a point on the function, the slope of the tangent line is equal to the derivative of the function at the same point.



(Graphic of tangent of f(x) at point (c,(f(c)) from https://brilliant.org/wiki/tangent-line-point/)

Thus, the tangent of a function f at the point *(a, f(a))* is given by f(a) + f′(a)(x-a)

Example:

Determine the tangent of f (x) = x^2 + x at the point (1, f (1)):

$$Equation\ of\ tangent:\ f(a) + f^{'}(a)(x - a)$$

Thus, as the question asks at point (1, (f(1)):

$$f(1) + f^{'}(1)(x - 1)$$

Solution:

First, we can find f(1) by substituting 1 to f(x):

$$f(1) = 1^2 + 1 = 2$$

Then, we can find f′(2) by differentiating f(x) and substituting 2 to the derivative of f(x):

$$f'(x) = 2x + 1$$
$$f'(1) = 2(1) + 1$$
$$f'(1) = 3$$

Then, we can substitute both results to the formula of tangent as written above:

$$f(1) + f'(1)(x - 1)$$
$$\Longrightarrow 2 + 3(x - 1)$$
$$\Longrightarrow 2 + 3x - 3$$
$$Tangent: 3x - 1$$

If *y* = *f*(*x*), then

$$f'(x) = \frac{df(x)}{dx} = \frac{dy}{dx} = y'$$

Simple rules for differentiation

Derivative of constant	$f(x) = c \Rightarrow f'(x) = 0$
Derivative of a power	$f(x) = x^P \Rightarrow f'(x) = Px^{P-1}$
Parameter rule	$cf(x) \Rightarrow cf'(x)$
Sum rule	$F(x) = f(x) + g(x)$ $\Rightarrow F'(x) = f'(x) + g'(x)$
Product formula	$F(x) = f(x)g(x)$ $\Rightarrow F'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient rule	$F(x) = \frac{f(x)}{g(x)}$ $\Rightarrow F'(x) = \frac{f(x)g'(x) - f'(x)g(x)}{(g(x))^2}$

Note:

- The properties may be used twice in a single function (for example, using both product rule & quotient rule) to differentiate the whole function.
- For quotient rule, the order of the formula (which part of the function is differentiated first) matters to the results, thus, follow the rule as you can!

Introduction to mathematics – IBEB – Lecture 4 – week 4
Advanced differentiation

Chain rule

The chain rule is a formula for computing the derivative of the composition of two or more functions. This is more formally stated because if the functions f (x) and g (x) are both differentiable and defined. The rule itself has two forms, as shown:

1. $\frac{df(g(x))}{dx} = \frac{da}{a}\frac{dg(x)}{g(x)}, \ where \ a = g(x)$

2. $\frac{df(g(x))}{dx}$ =f′(g(x))g′(x)

Where number 1 requires substitution and number 2 is a more direct approach of using the chain rule formula.

Example of using number 2:

Find the derivative of $f(x) = 3(x^3 + 1)^4$
In this case, we let:
$f(x) = 3(x^3 + 1)^4$ *and* $g(x) = x^3 + 1$
Formula: *f′(g(x))g′(x)*
Therefore, we can now find f′(x) and g(x):
$f'(x) = 12(x^3 + 1)^3$ *and* $g'(x) = 3x^2$

Therefore, the derivative of $f(x) = 3(x^3 + 1)^4$ *would be*:

$$f'(x) = 12\left(x^3 + 1\right)^3(3x^2)$$
$$\Rightarrow f'(x) = 36x^2\left(x^3 + 1\right)^3$$

Higher-order derivatives

Higher order derivatives are the derivative of a derivative, where the derivative of the derivative of *f* is denoted as a <u>second-order derivative</u> of f.

$$f''(x) = f^{(2)}(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d^{(2)}f(x)}{(dx)^2}$$

or,

$$f^{(n)}(x) = \frac{d^{(n)}f(x)}{(dx)^n}$$

Example:

Determine the third order derivative of $f(x) = x^4 + x^2$

The steps to find the third order derivative would be to differentiate the function 3 times (until third order), thus:

First order: $f'(x) = 4x^3 + 2x$

Second order: $f''(x) = 12x^2 + 2$

Third order: $f^{(3)}(x) = 24x$

* We can keep looking for derivatives of a derivative. We obtain the second derivative by deriving the first derivative, the third derivative by deriving the second derivative, and so on.

Theory increasing and decreasing

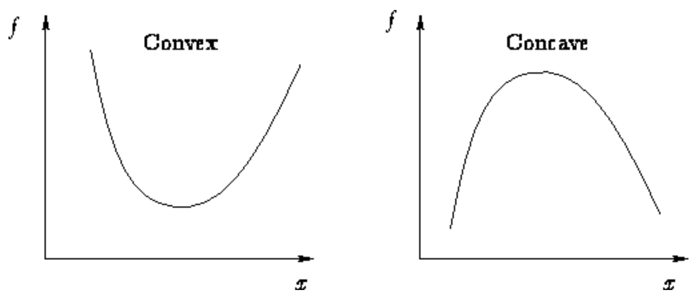
Using derivatives, we can determine exactly where the function increases and where it decreases. This is done by computing the function's derivative and make the **sign table** for f'(x). By making the sign table for f'(x) we actually compute the **variation table** of the function. Furthermore, by finding the signs of f'(x), we reach the following conclusions:

- If f'(a)>0, then f is **strictly** increasing in a
- If f'(a)<0, then f is **strictly** decreasing in a
- If f'(a)≥0, then f is increasing in a
- If f'(a)≤0, then f is decreasing in a

Convex & concave

Moreover, by using second-order derivatives, it can also be used to show or determine the shape of the function, and by using the sign table of the second derivative of f(x), we can actually determine where the function is **concave** and where it is **convex**, based on the sign of $f''(x)$, such that:

- If $f''(x) \geq 0$, then f is **convex** for all x in a specific interval and $f'(x)$ is **increasing.**
- If $f''(x) \leq 0$, then f is **concave** for all x in a specific interval and $f'(x)$ is **decreasing.**



(*https://math.stackexchange.com/questions/1208943/is-gx-log-x-convex-function*)

Increasing & decreasing: concave up & concave down

If $f'(x)>0$ and $f''(x)>0$, then f is concave up and increasing	The function $f(x)=x^3$ is concave up and increasing for $x>0$ because $f'(x)= 3x^2 >0$ $f''(x)= 6x>0$
If $f'(x)>0$ and $f''(x)<0$, then f is concave down and increasing	The function $f(x)=x^3$ is concave down and increasing for $x<0$ because $f'(x)= 3x^2 >0$ $f''(x)= 6x<0$
If $f'(x)<0$ and $f''(x)<0$, then f is concave down and decreasing	The function $f(x)=-x^5$ is concave down and decreasing for $x>0$ because $f'(x) =-5x4<0$ $f''(x) = -20x3< 0$
If $f'(x)<0$ and $f''(x)>0$, then f is concave up and decreasing	The function $f(x)=-x^5$ is concave down and decreasing for $x<0$ because $f'(x) =-5x^4<0$ $f''(x) = -20x^3 >0$

Derivatives of special functions

Logarithmic differentiation

Determining the derivative of a function can sometimes be difficult, so the logarithmic differentiation method is preferred instead. Thus, instead of "trying" to compute f'(x), we simply compute ln'(f(x))= $\frac{f'(x)}{f(x)}$, or the formula of:

$$\frac{df(x)}{dx} = f(x)\frac{d\, lnf(x)}{dx}$$

Logarithmic functions

For the natural logarithm ln(x), it holds that:

$$f(x) = ln(x) \Rightarrow f^{'}(x) = \frac{1}{x}$$

In order to find the derivatives of functions like ln(f(x)), we simply use the chain rule, since ln(f(x)) represents a composition of functions.

Example:

Find the derivative of $f(x) = (ln(x))^2$

We can see that it requires the usage of chain rule as well as the rule for logarithmic differentiation, therefore,

$$f^{'}(x) = 2(\ln ln\,(x)\,)\left(\frac{1}{x}\right)$$

$$f^{'}(x) = \left(\frac{2(\ln ln\,(x)\,)}{x}\right)$$

Derivatives of exponential functions

Derivative of exponential functions with "x" in the exponent

Now that we've established the natural logarithm, we may discuss exponential functions and their derivatives. Additionally, we can now compute the logarithm's derivative.

The formula of finding derivative functions is $y'(a^x) = \ln(a) \cdot a^x$

Example: Find the derivative of $y'(45^x)$:

$$y'(45^x)=\ln(45)\cdot 45^x$$

We can also calculate the derivative of $a^{g(x)}$ for a function $g(x)$ using the chain rule. This will give

$$y'(a^{g(x)})=\ln(a)\cdot g'(x)\cdot a^{g(x)}$$

Derivative of the base "e"

Exponential function's derivation rule is given:

$$f(x) = e^x \Rightarrow f'(x) = e^x$$

In order to find the derivatives of composed functions like e^{e^x}, the chain rule must be used. Denote e^x with u and derive the given function using the chain rule.
Note: The derivative of e^x is still e^x.

Derivative of the logarithm

The derivative of the logarithm is as follows: $y'(\log_a(x))=1 / x \cdot \ln(a)$

Example: Find the derivative of $y'(\log_8(x))$

$$y'(\log_8(x))=1 / x \cdot \ln(8)$$

Derivative of the natural logarithm

Below is the formula to obtain the derivative of a natural logarithm

$$y'(\ln(x)) = y'(\log e(x)) = 1/x\cdot\ln(e) = 1/x$$

Differentiation of inverse functions

As stated previously, an **inverse function** (or anti-function) is a function that "reverses" another function: *if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, and vice versa.*

To begin, we'll look at a function and its inverse. If f(x) is both invertible and differentiable, the inverse of f(x) should likewise be differentiable.

We can also find the derivative of the inverse.

Therefore, the derivative of g(x) for f(x)≠0 is where g(x) is the inverse:

$$g^{'}(x) = \frac{1}{f'(g(x))}$$

Implicit differentiation

As functions are not always defined explicitly, implicit differentiation is used in computing the derivative of these implicit function.

We differentiate each side of an equation with two variables (typically x and y) using implicit differentiation, which is defined as considering one of the variables as a function of the other variable. This necessitates the application of the chain rule.

Note: The implicit differentiation is usually applied when a “y” is defined implicitly.

Example:
Find the derivative of y with respect to x, with y implicitly defined as:

$$y^{2} + 4x^{2} = 7 - x$$

From here, y can be considered as a function of x, (y=f(x)):

$$(f(x))^{2} + 4x^{2} = 7 - x$$

Now we can differentiate each side of the equation:

$$2(f(x))\left(f^{'}(x)\right) + 8x =- 1$$

From here, we can re-arrange the function so that f’(x) will be on one side of the equation:

$$f^{'}(x) = \frac{-8x-1}{2(f(x))}$$

Lastly, we can substitute the f(x) back to the original variable of y:

$$y' = \frac{-8x-1}{2y}$$

Elasticity

The **elasticity** of a function f(x) is the ratio of the relative change in the function's output with respect to the relative change in its input x or the relationship of change (as applied in various economic models such as in the price elasticity of demand, etc.). The elasticity of f(x) with respect to x gives approximately the percentage change in f(x) when x is increased or decreased. **We use derivatives to compute elasticity.**

The notation of elasticity is as written:

$$El_x f(x)$$

and is calculated with the following formula:

$$El_x f(x) = \frac{x}{f(x)} \frac{df(x)}{dx},$$

where f′(x) represents the derivative of f(x).

By using the definition of the derivative of a function (Newton Quotient) we also reach this formula:

$$(x) = \frac{x}{f(x)} \frac{f(x+h)-f(x)}{h} = \frac{x}{h} x \frac{f(x+h)-f(x)}{f(x)}$$

Example:

Compute the elasticity of x from the given function $f(x) = \frac{x-1}{x+1}$

Solution:

$$El_x f(x) = \frac{x}{f(x)} \frac{df(x)}{dx}$$

We can compute by putting the function into the formula:

$$El_x\left(\frac{x-1}{x+1}\right) = \frac{x(x+1)}{x-1} \bullet \frac{1(x+1)-1(x-1)}{(x+1)^2} \Rightarrow \frac{2x}{x^2-1}$$

Elasticity of logarithmic devices

For a given function f(x) we have:

$El_x f(x) = \frac{dln(f(x))}{dln(x)}$, which is equivalent to $El_x f(x) = \frac{(\ln ln\,(f(x))\,)'}{(lnx)'}$

Note:
This is a different way of computing, not a definition of elasticity (or in other words, is just another formula to compute elasticity).

Introduction to mathematics – IBEB – Lecture 5 – week 5
Limits and extrema

The expression:

$$f(x) = A$$

Represents the limit of f(x) when x is sufficiently closer to a, but not equal to a.
Notes:
When computing lim*x*→*a* *f* (*x*), we must consider values of *x* on both sides of *a* (one-sided limits)

One-sided limits

Whenever a real number x approaches a random parameter k (obviously different form ±∞), it can be either lower or higher than k. Therefore, we say:

– $f(x)$ is the left limit of f(x), when x approaches k but is smaller than k or in other words, the left limit is the value that f when x approaches *a* from below (x < a)

– $f(x)$ is the right limit of f(x), when x approaches k but is bigger than k or in other words, the right limit is the value that f when x approaches *a* from above (x > a)

Whenever$f(x) = f(x) = A$, then the limit of f(x) in k exists and it equals A, $(f(x) = A)$

Asymptotes

An asymptote is a line for which, as x tends to ± ∞, the distance between the line and the function to which the line is an asymptote tends to, but never reaches, 0. The function f(x) = e^x has a horizontal asymptote y=0; and f(x) = lnx has a vertical asymptote is x=0.

We can also have non-horizontal and non-vertical asymptotes. If f(x) is a function of the form $f(x) = \frac{A(x)}{B(x)}$ where A(x) is *exactly* one degree greater (the largest power of x is 1 larger) than B(x), we will have an asymptote of the form y = ax +b. We can find this asymptote through polynomial division, because the remainder of the fraction will eventually tend to zero as x tends to ± ∞, leaving only the linear equation, which is the asymptote.

Horizontal asymptotes

To find the horizontal asymptote, we must compare the degree of the numerator "n" to the degree of the denominator "s"

If n < s, then:	The horizontal asymptote is y = 0
If n = s, then:	Divide the leading coefficients to find the horizontal asymptote.
If n > s, then:	There is no horizontal asymptote

Continuity and differentiability

Continuity

A graph is referred to continuous if the graph of a function is smooth and free of holes, leaps, or asymptotes.

A function f is **continuous** if for any given value y, that exists in the domain of f, $f(x) = f(y)$. Furthermore, a function is continuous in a point z from the domain of f if $f(x) = f(z)$.

In addition, it can be said that a function f is continuous in a given point g if $f(x)\, f(x) \; = f(g)$.

Important note:

- The function f is continuous in a if $f(x) = f(a)$

- The function f is continuous in a if $f(x) = f(a)$ for all a in the domain of f
- The function f is left continuous in a if $f(x) \ = f(a)$
- The function f is right continuous in a if $f(x) \ = f(a)$

Example: $f(x) = \{3x - 2,\ for\ x < 1\ 1,\ for\ x = 1\ - x + 2,\ for\ x > 1$

Is f continuous?

By applying the definition, we get $(3x - 2)\ (- x + 2) \ = f(1) = 1$. *This means that f is continuous in 1. Moreover, since f is composed of* ***elementary functions****, that are* ***always continuous****, then f is continuous in all its points.*

Differentiability

"If this limit exists, we say that *f* is *differentiable* at *x*. The process of finding the derivative of a function is called *differentiation*." (Knut Sydsæter, 2016). Therefore, we can conclude that:

- The function f is differentiable in a if $\frac{f(a+h)-f(a)}{h}$ exists.
- The function f is differentiable if $\frac{f(a+h)-f(a)}{h}$ exists for all a in the domain of f.
- The differentiability of a function implies its continuity. However, the reverse statement is not true.

L'Hopital

L'Hopital's rule is used to calculate the limits of functions that can be rearranged as a fraction and that are differentiable in a point k, as well as in all the values close to k.

Consider the functions f(x) and g(x) that are differentiable in and for the values of x close to a, however, if f(a) = g(a) = 0, then it is not clear what the limit is, as shown below:

$$\frac{f(x)}{g(x)} \ = \frac{0}{0}$$

The previously stated affirmations make sense if $\frac{f^{'}(x)}{g^{'}(x)}$ exists. Moreover, L'Hopital's rule can be applied even when k=±∞, as rule says that if $\frac{f^{'}(x)}{g^{'}(x)}$ exist,

$$\frac{f^{'}(x)}{g^{'}(x)}$$

In which the L'Hopital's rule can be done multiple times until higher order derivatives to get the limit of the original function. However, as soon as the result of the derivative is a zero or a number, we must stop as the answer is no longer an indeterminate form and the rule no longer applies.

Stationary points

A stationary point x of a differentiable function f satisfies the equation f'(x)=0.
There are three sorts of stationary points: maximum points, minimum points, and points of inflection. A stationary point is a point on a curve where the slope is zero.

Note:
Any local extreme point that is not an endpoint of the domain must be a stationary point.

Minima & maxima

Extrema

Extreme points are defined as values of x for which the function f reaches either its maximum or its minimum.

Global maximum and global minimum points

Maximum or minimum values that are reached across the whole function are referred to as "Absolute" or "Global" maximum or minimum values.

Unlike the global maximum (and minimum), which are both fixed, the local maximum and minimum might be many times larger or smaller.

Global Maximum: If for a point k it holds that f(k)≥f(x) for all the values of x in the function f's domain, then k is a global maximum point and f(k) is a global maximum.

Global Minimum: If for a point k it holds that f(k)≤ f(x) for all the values of x in the function f's domain, then k is a global minimum point and f(k) is a global minimum.

Local maximum and local minimum points

If for a point *a* it holds that f (a) is greater (less) than or equal to f (x) for all x in a neighbourhood of a, then we call a a local maximum (minimum) point, whereas f (a) is called a local maximum (minimum).

Note:
Every global maximum (minimum) is also a local maximum (minimum).

Extrema on a closed interval

A continuous function with domain [a,b] always has a maximum and minimum. The potential extreme points of a closed interval include:

1. All stationary points on (a, b), i.e., x in (a, b) such that f ′(x) = 0.
2. The point a
3. The point b

How to find the extreme of a differentiable function on a closed bounded interval

1. Find every stationary point on (a,b) and compute the value of f at these points.
2. Compute f's value in the endpoints of the interval
3. The function's maximum is the largest value found under numbers 1 and 2, and the smallest value is the minimum.

Second derivative & inflection points

Inflection points

Inflection points are a point of a curve at which a change in the direction of curvature occurs. Consider a function f, with f′′(k)=0, then k is an **inflection point** of f ***if*** f′′(x) changes sign at k.

To determine the inflection points of a function, we simply find the second derivative of the function (if it exists) and then solve the equation f′′(x)=0.

First derivative test

k can be considered a local extreme point of f(x), such that f′(k)=0 and if f′(x) changes sign at x=k.

Furthermore, by applying the afore-mentioned definition of local extreme points, it can be determined if k is a maximum local point or a minimum local point. If f′(x) does not change its sign at k, then k is not a local extreme point of the function.

Second derivative test

Let f be a twice differentiable function, with a stationary point k, such that f′(k)=0.
We have the following conditions:

1. If f′′(k)<0, then k is a local maximum point
2. If f′′(k)>0, then k is a local minimum point

If f′′(k)=0, then there is nothing that can be said about k. (we have to use the first derivative test).

Introduction to mathematics – IBEB – Lecture 6 – week 6
Multiple Variables

In a function of two variables, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to the values of another variable.

Partial differentiation

Functions of two or more variables (such as z=f(x,y)) have multiple independent, or exogenous, variables (in this case x and y), and a dependent, endogenous, variable (in this case z). The domain is usually defined as a combination the independent variables, i.e. there is often not a unique value that the domain takes.

Partial differentiation is the derivative of f with respect to a specific variable. Consider, $f(x_1, x_2, x_3, ...x_n)$ is a function with n variables summarized by x, the derivative of f with respect to x_n is called a **partial derivative**.

This is denoted by:

$$\frac{\partial y}{\partial x} = f'_{x_i}(x) = f'_i(x)$$

When computing partial differentiation, the other variable(s) in which we are not deriving with respect to remains as a constant(s).

Second-order partial derivatives

The partial derivative of a function of variables is a function of the variables that make up the function. We can compute the higher-order derivatives by taking the partial derivatives of the partial derivatives and multiplying them together.

The second derivative of a function f is calculated with respect to x_i first and then with respect to x_j, which is denoted as:

$$\frac{\partial f}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i} = f''_{x_i x_j}(x) = f''_{ij}(x)$$

Young's Theorem:

$f''_{ij}(x) = f''_{ji}(x)$ (for most well behaving functions), which shows that the order of the derivative of respecting different variables will not matter as the result will most likely be the same for all functions found in this course.

Hessian Matrix:

Hessian matrix is the summary of order partial derivatives systematically.

$$f''(x) = \left[f''_{11}(x) f''_{12}(x) \ \cdots \ f''_{1n}(x) \vdots \vdots \ddots \vdots \ f''_{n1}(x) f''_{n2}(x) \ \cdots \ f''_{nn}(x) \right]$$

According to Young's Theorem, this matrix will be symmetrical.

Note:

- It is very important to notice and/or realize which variable we are respecting to when calculating first or second order partial derivatives and ensuring that the other variables remain as constants.
- Young's Theorem should be recognized in order to not repeat calculations when computing second order partial derivatives that will have the same results.

Partial elasticity

The partial elasticity of the function z(a,b) with respect to the variable a is denoted by ϵa=z'a(a,b) · az(a,b). The partial elasticity of the function z(a,b) with respect to the variable y is denoted by ϵy=z'b(a,b) · bz(a,b).

Let us consider f(x) a function with n variables.
The partial elasticity of the function f with respect to a specific variable x_i is:

$$El_{x_i} f(x) = \frac{x_i}{f(x)} \frac{\partial f(x)}{\partial x_i}$$

$$\textit{where } \frac{\partial f(x)}{\partial x_i} \textit{ is the partial derivative in respect to } x_i$$

Or in terms of natural logarithm:

$$El_{x_i} f(x) = \frac{\partial ln(f(x))}{\partial ln(x_i)}$$

When computing the partial elasticity of a function, the previously mentioned rules apply. Furthermore, to simplify the calculations for the logarithmic form of the elasticity, the notation *ln(x)=u* is used. This way, we can make the substitution $e^u = x$.

Example:

$$f(x, y) = yx^x 2^{yln(x)}$$

We can now use the natural logarithm form of the partial elasticity formula, thus, we can input the function f(x,y) as:

$$\ln ln \left(yx^x 2^{yln(x)}\right)$$

$$\Rightarrow \ln ln\,(y) \; + \; xln(x) + \; yln(x)ln(2)$$

We can now let ln(x)=u, substituting x=e^u

$$El_x f(x) = \frac{\partial l(\ln ln\,(y) + e^u u + yuln(2))}{\partial u}$$

$$El_x f(x) = \frac{e^u u + e^u + y ln(2)}{1}$$
$$El_x f(x) = xln(x) + x + yln(2)$$

Chain rule

Simple chain rule

Considering the function $z = f(x, y)$ with $x = f(t)$ and $y = g(t)$, we have the following relationship when computing chain rule for functions with more than one variable:

$$\frac{\partial z}{\partial t} = \left(\frac{\partial F}{\partial x}\right)\frac{\partial x}{\partial t} + \left(\frac{\partial F}{\partial y}\right)\frac{\partial y}{\partial t}$$

If $z = f(x, y)$ and we consider n functions x_i of t, with i=1,2,3,...n $(x_i = f_i(x_1, x_2, x_3, ... x_n))$. We have the following relationship:

$$\frac{\partial z}{\partial t_i} = \left(\frac{\partial F}{\partial x_1}\right)\frac{\partial x_1}{\partial t_i} + \left(\frac{\partial F}{\partial x_2}\right)\frac{\partial x_2}{\partial t_i} + ... + \left(\frac{\partial F}{\partial x_n}\right)\frac{\partial x_n}{\partial t_i}$$

Alternative notation: $\frac{\partial z}{\partial t_i} = F_{t_i}(x)$

Implicit differentiation

A level set or level curve function or in other words, functions f with real value. If F(x,y) is a level curve defined by F(x,y)=c, then the differentiation of the function can be computed by:

$$y' = \frac{dy}{dx} = -\frac{F'(x)}{F'(y)}$$

Where $f'(x)$ is the partial differentiation in respect to x, and $f'(y)$ is the partial differentiation in respect to y.

Example:

Consider the function $F(x, y) = x^2 y$ and $c = 1$, compute the differentiation in respect to x.

Solution:

$$y' = \frac{dy}{dx} = -\frac{F'(x)}{F'(y)}$$

We can now compute the partial differentiation in respect to x and in respect to y, and us them inside the formula as shown above:

$$y' = -\frac{2xy}{x^2}$$

$$y' = -\frac{2y}{x}$$

Homogeneous functions

Homogenous functions are those that have multiplicative scaling behaviour in mathematics, meaning that if all of their arguments are multiplied by a factor, then the value of their function is multiplied by some power of the factor that multiplied all of their arguments.

If f(**x**) is a function with n variables, we can say that the function is **homogenous** of degree k if it holds that:

$$f(tx_1, tx_2, tx_3, ...tx_n) = t^k f(x_1, x_2, x_3, ...x_n)$$

How to determine whether the function is homogeneous or not:

- We multiply every variable with t
- Simplify when needed, and see whether or not the function comes back to the original function with t.

Properties

Let us consider a function F(x) that is homogenous of degree k.

1. **Euler's theorem**

$$x_1\left(\frac{\partial F}{\partial x_1}\right) + x_2\left(\frac{\partial F}{\partial x_2}\right) + \ ... \ + x_n\left(\frac{\partial F}{\partial x_n}\right) = aF(x)$$

2. **Partial derivative's degree**

The partial derivative $\frac{\partial F}{\partial x_i}$ of a function is homogeneous, with its degree being k-1.

To determine whether or not a function is homogeneous of degree k either the definition or Euler's theorem can be used.

Introduction to mathematics – IBEB – Lecture 7 – week 7
Optimization in two variables

Stationary point

For a function of two variables, such as f(x,y), if the point (ˆx, yˆ) is an *interior* point (i.e. it isn't on the boundary of the domain), and z=f(x,y) is a differentiable function, then the necessary condition for it to be an extreme point is:

$$f'_x(\hat{x}, \hat{y}) = 0$$
$$f'_y(\hat{x}, \hat{y}) = 0$$

A stationary point is always an interior point of the domain of the given function, in this case f.

There are three types of stationary points, namely:

1. Maximum Points: when (from positive values) the value of the derivative becomes negative after plugging in numbers after the maximum point.

2. Minimum Points: when (from negative values) the value of the derivative becomes positive after plugging in numbers after the maximum point.

3. Inflection Points: A graph's inflection point is where the graph's rise or decline type shifts. It is only possible for these points to exist if the second derivative is zero. The graph can change in the following ways:
   a. from concave up and increasing to concave down and increasing
   b. from concave down and increasing to concave up and increasing
   c. from concave down and decreasing to concave up and decreasing
   d. from concave up and decreasing to concave down and decreasing

Sufficient conditions with regard to the convexity or concavity of a function

Convexity of domain

The domain of a function is convex if for every pair of points within the specified region, every point on the straight line segment that connects them is also part of the region.

Convexity of function

A function f (x, y) is convex if any straight line segment connecting two points of the function's graph is situated completely on or above the graph.

Conditions for a function to be convex

For any function f(x, y) that is twice differentiable and of which the first and second derivative are continuous ($f \in C^2$) we can say that it is convex for all (x, y) in the domain if:

$$f''_{11}(x, y) \geq 0$$
$$f''_{22}(x, y) \geq 0$$
$$f''_{11}(x, y)\, f''_{22}(x, y) - (f''_{12}(x, y))^2 \geq 0$$

Concavity of function

A function f (x, y) is convex if any straight line segment connecting two points of the function's graph is situated completely on or below the graph.

Conditions for a function to be concave

For any function f(x, y) that is twice differentiable and of which the first and second derivative are continuous (f∈C2) we can admit that it is concave for all (x, y) in the domain if:

$$f''_{11}(x, y) \leq 0$$

$f''_{22}(x, y) \leq 0$ f

$''_{11}(x, y)\ f''_{22}(x, y) - (f''_{12}(x, y))^2 \geq 0$

Sufficient conditions for minimum

If we consider the function f (x, y), with f∈ C^2 and the point (ˆx, yˆ) we say that (ˆx, yˆ) is a minimum point if it respects the following conditions:

- the point (ˆx, yˆ) represents a stationary point of the function f.
- the domain of f is convex
- the function f is convex.

Sufficient conditions for maximum

If we consider the function f (x, y), with f∈ C2 and the point (ˆx, yˆ) we say that (ˆx, yˆ) is a maximum point if it respects the following conditions:

- the point (ˆx, yˆ) represents a stationary point of the function f.
- the domain of f is convex
- the function f is concave.

Local extrema of functions in two variables

By considering the function f (x, y) and the point (ˆx, yˆ) we can denote that:

- if f(x, y) ≤ f(ˆx, yˆ), for all the pairs (x, y) in the neighbourhood of (ˆx, yˆ), the point (ˆx, yˆ) is a local maximum.
- if f(x, y) ≥ f(ˆx, yˆ), for all the pairs (x, y) in the neighbourhood of (ˆx, yˆ), the point (ˆx, yˆ) is a local minimum.

Suppose that f is differentiable and the point (ˆx, yˆ) is an interior point of the domain of f. If (ˆx, yˆ) is a stationary point, then it is also a local extreme point (local minimum or maximum point).

Sufficient conditions local extrema

By considering the function f (x, y) and the point (ˆx, yˆ) as an interior pointof the domain of f, we have to consider the following situations:

Local maximum point

The point $(\hat{x}, \hat{y})$ is a local maximum point if:

$$f''_{11}(\hat{x}, \hat{y}) < 0$$
$$f''_{22}(\hat{x}, \hat{y}) < 0$$
$$f''_{11}(\hat{x}, \hat{y})\, f''_{22}(\hat{x}, \hat{y}) - (f''_{12}(\hat{x}, \hat{y}))^2 > 0$$

Local minimum point

The point $(\hat{x}, \hat{y})$ is a local minimum point if:

$$f''_{11}(\hat{x}, \hat{y}) > 0$$
$$f''_{22}(\hat{x}, \hat{y}) > 0$$
$$f''_{11}(\hat{x}, \hat{y})\, f''_{22}(\hat{x}, \hat{y}) - (f''_{12}(\hat{x}, \hat{y}))^2 > 0$$

Saddle point

The point $(\hat{x}, \hat{y})$ is a saddle point if:

$$f''_{11}(\hat{x}, \hat{y})\, f''_{22}(\hat{x}, \hat{y}) - (f''_{12}(\hat{x}, \hat{y}))^2 < 0$$

Inconclusive

The point $(\hat{x}, \hat{y})$ can be either a local maximum, local minimum or a saddle point if:

$$f''_{11}(\hat{x}, \hat{y})\, f''_{22}(\hat{x}, \hat{y}) - (f''_{12}(\hat{x}, \hat{y}))^2 = 0$$

Optimisation in unbounded closed domain and Lagrange

Lagrange multipliers are an approach for determining the local maximum and minimum of a function that is subject to equality requirements/constraints (i.e, under the constraint that one or more equations must be exactly fulfilled by the values of the variables that have been chosen).

If we consider the function $f(x, y)$ and $g(x, y) = c$ a constraint, the **Lagrange function** is defined as:

$$L(x, y) = f(x, y) - \lambda(g(x, y) - c))$$

Note that λ is a constant.

The point (x, y) is a considered a stationary point of the function L(x, y) if for some value of λ we have:

$$L_x(x, y) = f'_x(x, y) - \lambda g'_x(x, y) = 0$$
$$L_y(x, y) = f'_y(x, y) - \lambda g'_y(x, y) = 0$$
with $g(x, y) = c$

The equations noted above are considered the **first-order conditions of Lagrange function.**

Considering:

$$\min f(x, y)$$
$$\text{with } g(x, y) = c$$

We imply that we want to determine the minimum of f(x, y) with domain defined by g(x, y) = c.

The minimum point $(\hat{x}, \hat{y})$ represents a stationary point of the Lagrange function. The maximum point $(\hat{x}, \hat{y})$ is also considered a stationary point of the Lagrange function.

Sufficient conditions optimisation with a constraint

We consider the following function defined in regard to x and y:

$$D = (f''_{xx} - \lambda g''_{xx})(g'_y)^2 - 2(f''_{xy} - \lambda g''_{xy}) \cdot g'_x \cdot g'_y + (f''_{yy} - \lambda g''_{yy})(g'_x)^2$$

Local minimum point

The point $(\hat{x}, \hat{y})$ is considered a local maximum point for:

- if $(\hat{x}, \hat{y})$ is a stationary point of the Lagrange function with $\lambda = \hat{\lambda}$.
- if $D(\hat{x}, \hat{y}) < 0$, where λ is taken equal to $\hat{\lambda}$.

Local minimum point

The point $(\hat{x}, \hat{y})$ is considered a local maximum point for:

- if $(\hat{x}, \hat{y})$ is a stationary point of the Lagrange function with $\lambda = \hat{\lambda}$

- if D(ˆx, yˆ) > 0, where λ is taken equal to λˆ.

Interpretation of λ

In the case of (ˆx, yˆ) being an extreme point of the function f(x, y) in regard to the constraint g(x, y) = c, we can observe that the extreme point depends on c (ˆx(c), yˆ(c)). Moreover, the extreme value also depends on c, fˆ(c).

The constant λ is expressed as follows:

$$\lambda = \frac{df^{\wedge}(c)}{dc}$$

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